CHARACTERIZATIONS OF SOME NEW CLASSES OF FUZZY SETS IN GENERALIZED FUZZY TOPOLOGY

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Abstract. In the present paper we introduce the concepts of maximal $\mu f$-open sets, minimal $\mu f$-closed sets, local minimal $\mu f$-open sets etc. in a generalized fuzzy topological space. We study their fundamental properties and discuss relations among these different $\mu f$-open like sets.

1. Introduction and Preliminaries

After the foundation of fuzzy sets by L. A. Zadeh [10], its multidirectional applications in different branches of modern science inspired Chang [1] to introduce the concept of fuzzy topology which is a generalization of classical set topology. Further generalization was contemplated by Chetty [2], who introduced generalized fuzzy topology. In this paper, we introduce a few new classes of fuzzy sets, termed maximal $\mu f$-open sets, minimal $\mu f$-closed sets in Section 2, and discuss their behaviors in different situations in a generalized fuzzy topological space. In Section 3, we define local minimal $\mu f$-open sets at some point $x$ of a non-empty set $X$ as well as at some fuzzy point $x_{\lambda}$ defined on $X$. Several results are obtained while discussing their properties and inter-relations.

2000 Mathematics Subject Classification. Primary 54A40, secondary 54D10, 54D15.

Key words and phrases. Generalized fuzzy topology, maximal $\mu f$-open set, minimal $\mu f$-open set, local minimal $\mu f$-open set, fuzzy $(\mu, \lambda)$-continuous function.

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Received: Feb. 19, 2015 , Accepted: Aug. 9, 2015 .
A fuzzy set \( A \) in \( X \) is characterized by a membership function in the sense of Zadeh [10]. The basic fuzzy sets are the zero set, the whole set and the class of all fuzzy sets in \( X \), to be denoted by \( 0_X \) and \( 1_X \) and \( I^X \) respectively. According to Chetty [2], a subcollection \( \mu \) of \( I^X \) is called a generalized fuzzy topology (GFT, for short) if \( 0_X \in \mu \) and \( \mu \) is closed under arbitrary unions of the members of \( \mu \). The structure \((X, \mu)\), where \( X \) is a non-empty set and \( \mu \) is a generalized fuzzy topology defined on \( X \), is said to be a generalized fuzzy topological space (GFTS, for short).

In what follows, by \((X, \mu)\) or simply \( X \) we will mean a GFTS. The members of \( \mu \) are called \( \mu \)-open sets and their complements are said to be \( \mu \)-closed sets. For any \( A \in I^X \), the \( \mu \)-closure of \( A \) and \( \mu \)-interior of \( A \) are denoted by \( c_\mu(A) \) and \( i_\mu(A) \) respectively and are defined by \( c_\mu(A) = \bigwedge \{ F : A \leq F, F \text{ is } \mu \text{-closed} \} \) and \( i_\mu(A) = \bigvee \{ V \in \mu : V \leq A \} \). For any two fuzzy sets \( A, B \) in \( X \), we write \( A \leq B \) if \( A(x) \leq B(x) \), for each \( x \in X \) whereas if \( A \leq B \) and \( A(x) \neq B(x) \) for some \( x \in X \) we write \( A < B \). The notation \( AqB \) means that \( A \) is quasi-coincident [7] with \( B \), i.e., \( AqB \), if \( A(x) + B(x) > 1 \) for some \( x \in X \). The negation of this statement is denoted by \( A\overline{q}B \). For a fuzzy set \( A \) in \( X \), the support of \( A \), denoted by \( S(A) \), is defined by \( S(A) = \{ x \in X : A(x) > 0 \} \) [10]. The union \( \bigvee A_\alpha \) and intersection \( \bigwedge A_\alpha \) of a family \( \{ A_\alpha : \alpha \in A \} \) of fuzzy sets \( A_\alpha \) are defined in the usual way (see [10]). A fuzzy singleton or a fuzzy point [7] with support \( x \) and value \( \alpha \) \((0 < \alpha \leq 1)\) is denoted by \( x_\alpha \). The fuzzy complement of a fuzzy set \( A \) in an GFTS \( X \), is written as \( 1 - A \). For a crisp set \( A \) of \( X \), \( \chi_A \) will stand for the characteristic function of \( A \), and the cardinality of any set \( Y \) will be denoted by \( |Y| \).

2. Maximal \( \mu \)-open sets and minimal \( \mu \)-closed sets

In this section, we introduce and investigate maximal \( \mu \)-open sets and minimal \( \mu \)-closed sets.
Definition 2.1. Let \((X, \mu)\) be a GFTS. A fuzzy \(\mu\)-open set \(A\) in \(X\) \((A \neq \mathbb{1}_X)\) is said to be a fuzzy maximal \(\mu\)-open set (maximal \(\mu f\)-open set, for short) in \(X\) if for any \(B \in \mu, \ (A \leq B \Rightarrow \text{either } B = A \text{ or } B = \mathbb{1}_X)\). The set of all maximal \(\mu f\)-open sets in \((X, \mu)\) is denoted by \(\text{max}(X, \mu)\).

Example 2.1. Let \(X = \{a, b, c\}\) and \(\mu = \{0_X, A, B, A \lor B\}\) be a GFT on \(X\), where \(A(a) = 0.2, A(b) = 0.8, A(c) = 0.4\) and \(B(a) = 0.6, B(b) = 0.5, B(c) = 0.3\). Here \(A \lor B\) is a maximal \(\mu f\)-open set in \((X, \mu)\).

Definition 2.2. A fuzzy \(\mu\)-closed set \(F\) in a GFTS \((X, \mu)\) with \(F \neq \mathbb{1}_X\), is called a fuzzy minimal \(\mu\)-closed set or a minimal \(\mu f\)-closed set in \(X\) if there is no \(\mu f\)-closed set lying strictly between \(0_X\) and \(F\).

Example 2.2. Let \(X = \{a, b\}\) and \(\mu = \{0_X, A, B, A \lor B\}\), where \(A(a) = 0.3, A(b) = 0.5; B(a) = 0.2\) and \(B(b) = 0.6\), be a GFT on \(X\). It is clear that \((1 - A \lor B)\) is a minimal \(\mu f\)-closed set in \((X, \mu)\).

By definition of maximal \(\mu f\)-open set, it is clear that maximal \(\mu f\)-open sets are all \(\mu f\)-open, although the converse is not true, in general. We show this by the following example:

Example 2.3. Let \((X, \mu)\) be a GFTS, where \(X = \{a, b\}\), \(\mu = \{0_X, A, B, A \lor B\}\) such that \(A(a) = 0.4, A(b) = 0.6; B(a) = 0.2, B(b) = 0.8\). Clearly \(A\) and \(B\) are both \(\mu f\)-open sets but they are not maximal \(\mu f\)-open.

The following result gives a relation between maximal \(\mu f\)-open set and minimal \(\mu f\)-closed set.

Theorem 2.1. A non-null fuzzy set \(U(\neq \mathbb{1}_X)\) in a GFTS \((X, \mu)\) is maximal \(\mu f\)-open iff \((1 - U)\) is minimal \(\mu f\)-closed.

Proof. Let \(U\) be a maximal \(\mu f\)-open set in \((X, \mu)\) and let \(F\) be a \(\mu f\)-closed set such
that $F \leq (1-U)$. Then $U \leq (1-F) \in \mu$. Now $U$ being maximal $\mu f$-open, $(1-F)$ is either $U$ or $1_X$.

If $(1-F) = U$ then $F = (1-U)$, and if $(1-F) = 1_X$ then $F = 0_X$. Thus we conclude that $(1-U)$ is a minimal $\mu f$-closed set.

Conversely, let $B$ be a minimal $\mu f$-closed set and $G$ be any $\mu f$-open set such that $(1-B) \leq G$. Since $B$ is minimal $\mu f$-closed, $(1-G)$ is either $0_X$ or $B$. Now $(1-G) = 0_X \Rightarrow G = 1_X$, and $(1-G) = B \Rightarrow G = (1-B)$. Thus $(1-B)$ is a maximal $\mu f$-open set.

**Theorem 2.2.** Let $(X, \mu)$ be a GFTS and $A \in \max(X, \mu)$. If $B$ is a non-zero $\mu f$-open set such that $A \land B = 0_X$, then $A = \chi_{S(A)}$ and $B = 1 - A$.

**Proof.** Let $y \in S(B)$. Since $A \land B = 0_X$, $A(y) = 0$. Thus $(A \lor B)(y) = B(y) \neq A(y)$.

Since $A \in \max(X, \mu)$, $A \lor B = 1_X$. Again since $A \land B = 0_X$, it follows that $A = \chi_{S(A)}$ and $B = 1 - A$.

**Theorem 2.3.** Let $(X, \mu)$ be a GFTS. If $A = \chi_{S(A)} \in \max(X, \mu)$ then either $c_\mu(A) = A$ or $c_\mu(A) = 1_X$.

**Proof.** If $c_\mu(A) = 1_X$ then there is nothing to prove. So let $c_\mu(A) \neq 1_X$. Then there exists $y \in X$ such that $c_\mu(A)(y) < 1$. Let $B = 1 - c_\mu(A)$. Then $B \in \mu$, $B \neq 0_X$ and $A \land B = 0_X$. Hence by Theorem 2.2, $B = 1 - A$ which implies that $c_\mu(A) = A$.

**Theorem 2.4.** Let $(X, \mu)$ be a GFTS and $A \in \max(X, \mu)$. If $B$ is a non-null fuzzy set in $X$ with $B \leq 1 - A$ then $c_\mu(B) = 1 - A$.

**Proof.** If possible, let $c_\mu(B) \neq 1 - A$. Since $B \leq (1-A)$ where $A \in \mu$, we have $c_\mu(B) \leq (1-A)$. Again since $c_\mu(B) \neq (1-A)$, it follows that there exists $x_0 \in X$ such that $c_\mu(B)(x_0) < (1-A)(x_0)$. Now, $A \in \max(X, \mu)$ and $A(x_0) < 1 - c_\mu(B)(x_0) \Rightarrow (1 - c_\mu(B)) = 1_X \Rightarrow c_\mu(B) = 0_X$, a contradiction.

**Theorem 2.5.** Let $(X, \mu)$ be a GFTS and $A = \chi_{S(A)} \in \max(X, \mu)$. If $B$ is a $\mu f$-closed set in $X$ such that $A < B$, then $B = 1_X$. 
Proof. If possible, let there exist \( y \in X \) such that \( B(y) < 1 \). Then \( [(1 - B) \lor A](y) \neq 0 = A(y) \) (since \( y \in X \setminus S(A) \) as \( A < B \)). Hence \( A \neq 1_{X} \) and \( A = \chi_{S(A)}, S(A) \neq X \). Thus we can choose \( x_1 \in X \setminus S(A) \) and then \( (1 - B)(x_1) = 1 \). Since \( A < B \), \( A(x_1) < B(x_1) \). Now \( (1 - B)(x_1) = 1 \Rightarrow B(x_1) = 0 > A(x_1) \) which is a contradiction.

**Corollary 2.1.** Let \( (X, \mu) \) be a \( GFTS \) and \( A = \chi_{S(A)} \in \max(X, \mu) \). If \( B \in I^{X} \) such that \( A < B \), then \( c_{\mu}(B) = 1_{X} \).

**Theorem 2.6.** Let \( (X, \mu) \) be a \( GFTS \) and \( A \in I^{X} \) such that \( A \neq (1 - A) \). Then the following are equivalent:

(a) \( \{ A, (1 - A) \} \subseteq \max(X, \mu) \).

(b) \( A = \chi_{S(A)} \) and \( \mu = \{ 0_{X}, A, (1 - A), 1_{X} \} \).

Proof. (a) \( \Rightarrow \) (b) : Since \( A \neq (1 - A) \), we choose \( x_0 \in X \) such that \( A(x_0) \neq (1 - A)(x_0) \). Then \( (A \lor (1 - A))(x_0) \neq A(x_0) \) or \( (A \lor (1 - A))(x_0) \neq (1 - A)(x_0) \). So by (a), \( (A \lor (1 - A)) = 1_{X} \). Hence for every \( x \in X \) with \( A(x) < 1 \), we must have \( (1 - A)(x) = 1 \) and it follows that \( A(x) = 0 \Rightarrow A = \chi_{S(A)} \). Next let \( B \in \mu \setminus \{ 0_{X} \} \).

If \( B \leq A \), then \( (1 - A) \land B = 0_{X} \) and hence by Theorem 2.2, \( B = 1 - (1 - A) = A \).

If \( B \not\leq A \), i.e., there exists \( x_0 \in X \setminus S(A) \) such that \( B(x_0) > A(x_0) = 0 \); then \( A \lor B = 1_{X} \), as \( A \in \max(X, \mu) \). Thus \( B(x) = 1 \), for all \( x \in X \setminus S(A) \) ... (i)

Now three cases arise:

**Case 1:** Let \( B(x) = 1 \), for all \( x \in S(A) \); then \( B = 1_{X} \).

**Case 2:** Let \( B(x) = 0 \), for all \( x \in S(A) \); then \( B = 1 - A \).

**Case 3:** Let \( B(x) = t \), where \( 0 < t < 1 \), for some \( x \in S(A) \). Then \( (1 - A) \lor B > 1 - A \) and hence by maximality of \( (1 - A) \), \( (1 - A) \lor B = 1_{X} \). Thus \( B(x) = 1 = t \), a contradiction. Hence case(III) is not tenable, so that \( B = 1_{X} \) or \( 1 - A \).

(b) \( \Rightarrow \) (a): Obvious.
Theorem 2.7. Let \((X, \mu)\) be a GFTS. Then the following statements are true:

(a) If \(U\) is a maximal \(\mu_f\)-open set and \(V\) is any \(\mu_f\)-open set then either \(U \cap V = 1_X\) or \(V \subseteq U\).

(b) For any two maximal \(\mu_f\)-open sets \(U\) and \(V\), either \(U \cap V = 1_X\) or \(U = V\).

Proof. (a) Here two cases arise:

Case-I: \(U \cap V = 1_X\). In this case we get the result.

Case-II: \(U \cap V \neq 1_X\). Then \(U \cap V\) is a \(\mu_f\)-open set for which \(U \cap (U \cap V) = U\) (since \(U \cap V \neq 1_X\)).

(b) For two maximal \(\mu_f\)-open sets \(U\) and \(V\), either \(U \cap V = 1_X\) or \(U \cap V \neq 1_X\). If \(U \cap V \neq 1_X\), then \(U \cap V\) is a \(\mu_f\)-open set such that \(U, V \subseteq U \cap V \Rightarrow U \cap V = U = V\).

Corollary 2.2. Let \((X, \mu)\) be a GFTS and \(A \in \text{max}(X, \mu)\) with \(I(A) = \phi\), where \(I(A) = \{x \in X : A(x) = 1\}\). Then for every \(B \in \mu \setminus \{1_X\}\), \(B \leq A\).

Proof. Follows from Theorem 2.7(a).

Corollary 2.3. Let \((X, \mu)\) be a GFTS and \(A \in \text{max}(X, \mu)\) with \(I(A) = \phi\). Then \(\text{max}(X, \mu) = \{A\}\).

Corollary 2.4. Let \((X, \mu)\) be a GFTS. If \(|\text{max}(X, \mu)| > 1\), then for every \(A \in \text{max}(X, \mu)\), \(I(A) \neq \phi\).

Theorem 2.8. Let \((X, \mu)\) be a GFTS. Then the following statements are true:

(a) If \(F\) is a minimal \(\mu_f\)-closed set and \(G\) is any \(\mu_f\)-closed set then either \(F \cap G = 0_X\) or \(F \subseteq G\).

(b) For any two minimal \(\mu_f\)-closed sets \(F\) and \(G\), either \(F \cap G = 0_X\) or \(F = G\).

Proof. The proof is similar to that of Theorem 2.7.

Corollary 2.5. Let \((X, \mu)\) be a GFTS in which \(U\) is a maximal \(\mu_f\)-open set and \(x_\lambda\) a fuzzy point such that \(x_\lambda q(1 - U)\). Then for any \(\mu_f\)-open set \(V\) in \(X\) containing \(x_\lambda\), \((1 - U) \leq V\).
Proof. Since \( x_\lambda q(1 - U) \), then \( x_\lambda \not\in U \). Thus for any \( \mu f \)-open set \( V \) in \( X \) containing \( x_\lambda \), \( V \not\subseteq U \). Hence by Theorem 2.7(a), \( U \cup V = 1_X \Rightarrow (1 - U) \leq V \).

**Corollary 2.6.** For any maximal \( \mu f \)-open set \( U \) in any GFTS \( (X, \mu) \), only one of the following statements (a) and (b) holds:

(a) For each fuzzy point \( x_\lambda \) in \( X \), if \( x_\lambda q(1 - U) \) then for each \( \mu f \)-open set \( V \) in \( X \) containing \( x_\lambda \), \( V \not\subseteq U \).

(b) There exists a fuzzy point \( x_\lambda \) with \( x_\lambda q(1 - U) \) and there exists a \( \mu f \)-open set \( V \) in \( X \) containing \( x_\lambda \) such that \( (1 - U) \leq V \) and \( V \not= 1_X \).

**Proof.** If (a) holds, then we are done. On the other hand, if (a) does not hold then there exist a fuzzy point \( x_\lambda \) in \( X \) and a \( \mu f \)-open set \( V \) containing \( x_\lambda \) such that \( x_\lambda q(1 - U) \) and \( V \not\subseteq 1_X \). Clearly \( V < 1_X \). Then by Theorem 2.7(a), \( U \cup V = 1_X \) or \( V \subseteq U \). But \( x_\lambda q(1 - U) \Rightarrow V \not\subseteq U \). Thus \( U \cup V = 1_X \) and so \( (1 - U) \leq V \).

**Theorem 2.9.** Let \( C \) be a minimal \( \mu f \)-closed set in a GFTS \( (X, \mu) \) and \( x_\lambda \) be any fuzzy point in \( X \) such that \( x_\lambda \subseteq C \). Then

(i) \( C \subseteq F \) for any \( \mu f \)-closed set \( F \) containing \( x_\lambda \).

(ii) \( C = \bigwedge \{ F : x_\lambda \subseteq F \text{ and } F \text{ is } \mu f \text{-closed } \} \).

**Proof.** (i) Let \( x_\lambda \subseteq C \) and \( F \) be a \( \mu f \)-closed set such that \( x_\lambda \subseteq F \). Then \( C \cap F \neq 0_X \).

By Theorem 2.8(a), \( C \bigwedge F \).

(ii) By (i) above, \( C \bigwedge \bigwedge \{ F : x_\lambda \subseteq F \text{ and } F \text{ is } \mu f \text{-closed } \} \). On the other hand, \( y_\beta \subseteq \bigwedge \{ F : x_\lambda \subseteq F \text{ and } F \text{ is } \mu f \text{-closed } \} \Rightarrow y_\beta \subseteq F \), for all \( \mu f \)-closed set \( F \) containing \( x_\lambda \Rightarrow y_\beta \subseteq C \Rightarrow \bigwedge \{ F : x_\lambda \subseteq F \text{ and } F \text{ is } \mu f \text{-closed } \} = C \).

**Theorem 2.10.** Let \( \{ F_\alpha : \alpha \in \Lambda \} \) be a family of minimal \( \mu f \)-closed sets in a GFTS \( (X, \mu) \) and \( F \) be a minimal \( \mu f \)-closed set in \( X \).

(a) If \( \bigcup_{\alpha \in \Lambda} F_\alpha \), then there exists some \( \alpha_0 \in \Lambda \) such that \( F = F_{\alpha_0} \).

(b) If \( F \not= F_\alpha \) for any \( \alpha \in \Lambda \), then \( \bigwedge_{\alpha \in \Lambda} F_\alpha \cap F = 0_X \).
Proof. (a) We first show that $F \land F_{0} \neq 0_{X}$ for at least one $\alpha_{0} \in \Lambda$. If possible, let $F \land F_{\alpha} = 0_{X}$ for each $\alpha \in \Lambda$. Then $F_{\alpha} \subset F_{0}$ for each $\alpha \in \Lambda$ which implies $F_{\alpha} \leq (1 - F)$ for each $\alpha \in \Lambda \Rightarrow \bigvee_{\alpha \in \Lambda} F_{\alpha} \leq (1 - F) \Rightarrow F_{\alpha} = (1 - F)$, a contradiction. Thus $F \land F_{0} \neq 0_{X}$ for some $\alpha_{0} \in \Lambda$. Since $F$ and $F_{0}$ are both minimal $f$-closed sets in $X$, by Theorem 2.8(b), $F = F_{0}$.

(b) If possible, let $(\bigvee_{\alpha \in \Lambda} F_{\alpha}) \land F \neq 0_{X}$. Then there exists $\alpha \in \Lambda$ such that $F_{\alpha} \land F \neq 0_{X}$. Now by Theorem 2.8(b), $F = F_{\alpha}$ for that $\alpha$, which contradicts our assumption.

Let us recall the definitions of fuzzy $(\mu, \lambda)$-continuous and fuzzy $(\mu, \lambda)$-open functions which are defined in [5]. Our goal is to look for the behavior of maximal $\mu f$-open sets under these functions.

**Definition 2.3.** Let $(X, \mu)$ and $(Y, \lambda)$ be two GFTS’s. A mapping $f : (X, \mu) \rightarrow (Y, \lambda)$ is said to be

(i) fuzzy $(\mu, \lambda)$-continuous if $f^{-1}(F)$ is $\mu f$-closed for every $\lambda f$-closed set $F$ in $Y$.

(ii) fuzzy $(\mu, \lambda)$-open if for every $\mu f$-open set $U$ in $X$, $f(U)$ is $\lambda f$-open in $Y$.

**Theorem 2.11.** Let $(X, \mu)$ and $(Y, \lambda)$ be two GFTS’s and $f : (X, \mu) \rightarrow (Y, \lambda)$ be a fuzzy $(\mu, \lambda)$-continuous and fuzzy $(\mu, \lambda)$-open surjection. If $A \in \max(X, \mu)$ then either $f(A) = 1_{Y}$ or $f(A) \in \max(Y, \lambda)$.

**Proof.** If $f(A) = 1_{Y}$ then there is nothing to prove. So let $f(A) \neq 1_{Y}$. Since $f$ is fuzzy $(\mu, \lambda)$-open and $A$ is a $\mu f$-open set in $X$, $f(A)$ is $\lambda f$-open in $Y$. Again since $A \neq 0_{X}$, there exists $x_{0} \in X$ such that $A(x_{0}) > 0$ and so $f(A)(f(x_{0})) = \sup\{A(x) : f(x) = f(x_{0})\} \geq A(x_{0}) > 0$ and hence $f(A) \neq 0_{Y}$. Let $B \in \lambda$ such that $f(A) < B$. It is sufficient to show that $B = 1_{Y}$. Let us choose $y_{0} \in Y$ such that $f(A)(y_{0}) < B(y_{0})$. Since $f$ is surjective, there exists $x_{0} \in X$ such that $f(x_{0}) = y_{0}$. Thus $A(x_{0}) \leq f(A)(y_{0}) < B(y_{0})$. Now $f$ being fuzzy $(\mu, \lambda)$-continuous, $f^{-1}(B) \in \mu$. Hence we get $f^{-1}(B) \lor A \in \mu, A \leq f^{-1}(B) \lor A$ and
\((f^{-1}(B) \lor A)(x_0) = \max\{A(x_0), B(y_0)\} = B(y_0) > A(x_0)\). Since \(A \in \max(X, \mu)\),
\(f^{-1}(B) \lor A = 1_X\). Next let \(y \in Y\). Then there exists \(x \in X\) such that \(f(x) = y\).
Thus \(1 = \max\{A(x), B(y)\} \leq \max\{f(A)(y), B(y)\} = B(y)\) and so \(B(y) = 1\). Hence \(B = 1_Y\).

3. Local minimal \(\mu f\)-open sets

In this section, we develop the notion of locally minimal \(\mu f\)-open sets at some point of a non-empty set \(X\) as well as at some fuzzy point \(x_\lambda\), defined on a GFTS \(X\) and study their basic properties.

**Definition 3.1.** Let \((X, \mu)\) be a GFTS, \(x \in X\) and \(A \in \mu\) such that \(x \in S(A)\). Then \(A\) is called a locally minimal \(\mu f\)-open set at \(x\) if for each \(B \in \mu\) with \(x \in S(B)\) one has \(A \leq B\). The set of all locally minimal \(\mu f\)-open sets at a point \(x \in X\) is denoted by \(\min(X, \mu, x)\).

**Definition 3.2.** Let \((X, \mu)\) be a GFTS, \(p_\lambda\) a fuzzy point in \(X\) and \(A \in \mu\) such that \(p_\lambda \leq A\). Then \(A\) is called a locally minimal \(\mu f\)-open set at \(p_\lambda\) if for each \(B \in \mu\) with \(p_\lambda \leq B, A \leq B\) holds. The set of all locally minimal \(\mu f\)-open sets at \(p_\lambda\) will be denoted by \(\min(X, \mu, p_\lambda)\).

**Example 3.1.** Let \(X = \{a, b, c\}, \mu = \{0_X, P, Q\}\) be a GFT on \(X\), where \(P(a) = 0.2, P(b) = 0.4, P(c) = 0.6\) and \(Q(a) = 0.3, Q(b) = 0.6, Q(c) = 1\). Here \(P \in \mu\) is a locally minimal \(\mu f\)-open set at \(a \in X\). Now we consider a fuzzy point \(c_{0.7}\) in \(X\). Then \(c_{0.7} \leq Q \in \mu\) and clearly \(Q\) is a locally minimal \(\mu f\)-open set at \(c_{0.7}\).

**Theorem 3.1.** Let \((X, \mu)\) be a GFTS, \(x \in X\) and \(p_\lambda\) be any fuzzy point in \(X\). Then 
\(|\min(X, \mu, x)| \leq 1\) and 
\(|\min(X, \mu, p_\lambda)| \leq 1\).

**Proof.** Let \(A, B \in \min(X, \mu, x)\). Then by definition, we have \(A \leq B\) and \(B \leq A\) and hence \(A = B\). Therefore \(|\min(X, \mu, x)| \leq 1\). Similarly we can show that 
\(|\min(X, \mu, p_\lambda)| \leq 1\).
Theorem 3.2. Let \((X, \mu)\) be a GFTS, \(A \in \mu\) and \(x \in X\). Then the following are equivalent:

(a) \(\min(X, \mu, x) = \{A\}\).

(b) For each fuzzy point \(x_\lambda \leq A\), \(\min(X, \mu, x_\lambda) = \{A\}\).

Proof. (a) \(\Rightarrow\) (b): Let \(x_\lambda\) be a fuzzy point in \(X\) such that \(x_\lambda \leq A\). Let \(B \in \mu\) such that \(x_\lambda \leq B\). Then \(x \in S(B)\). Since \(\min(X, \mu, x) = \{A\}\), we have \(A \leq B\). So \(\min(X, \mu, x_\lambda) = \{A\}\).

(b) \(\Rightarrow\) (a): Let \(B \in \mu\) such that \(x \in S(B)\). Let us consider a fuzzy point \(x_\lambda\) in \(X\) where \(\lambda = \min\{\frac{A(x)}{2}, \frac{B(x)}{2}\}\). Then \(x_\lambda \leq A \wedge B\). Since by (b), \(\min(X, \mu, x_\lambda) = \{A\}\), we have \(A \leq B\). So \(\min(X, \mu, x) = \{A\}\).

Theorem 3.3. Let \((X, \mu)\) be a GFTS, \(A \in \mu\) and \(p_\lambda\) be any fuzzy point in \(X\) with \(p_\lambda \leq A\). Then the following are equivalent:

(a) \(\min(X, \mu, p_\lambda) = \{A\}\).

(b) \(\forall B \in \mu : p_\lambda \leq B\) \(\Rightarrow\) \(\min(X, \mu, x_\lambda) = \{A\}\), for every fuzzy point \(p_\beta \leq A\) with \(\lambda \leq \beta\).

Proof. (a) \(\Rightarrow\) (b): Suppose that \(p_\beta\) is a fuzzy point with \(p_\beta \leq A\) and \(\lambda \leq \beta\) hold. Let \(B \in \mu\) with \(p_\beta \leq B\). Then \(\beta \leq B(p)\) and \(\lambda \leq \beta \leq B(p) \Rightarrow p_\lambda \leq B\). Then by (a), it follows that \(A \leq B\) and hence \(\min(X, \mu, p_\beta) = \{A\}\).

(b) \(\Rightarrow\) (a): Clear.

Theorem 3.4. Let \((X, \mu)\) be a GFTS and \(p_\lambda\) be any fuzzy point in \(X\). Then the following are equivalent:

(a) \(\min(X, \mu, p_\lambda) \neq \emptyset\).

(b) \(\forall B \in \mu : p_\lambda \leq B\) \(\in \mu\).

Proof. (a) \(\Rightarrow\) (b): In view of Theorem 3.1 and (a), we have \(\min(X, \mu, p_\lambda) = \{A\}\) for some fuzzy set \(A\). Now, for each \(B \in \mu\) with \(p_\lambda \leq B\) we have \(A \leq B\). Thus \(A \leq \bigwedge\{B \in \mu : p_\lambda \leq B\}\). Also by definition of \(\min(X, \mu, p_\lambda)\), \(p_\lambda \leq A(\in \mu)\). Hence \(\bigwedge\{B \in \mu : p_\lambda \leq B\} \leq A\). Therefore \(\bigwedge\{B \in \mu : p_\lambda \leq B\} = A \in \mu\).
Remark 1. The counterpart of the above theorem for the locally minimal set \(0\), viz ‘ for any \(x \in X\), where \((X, \mu)\) is a GFTS, \(\min(X, \mu, x) \neq \phi\) iff \(\bigwedge\{B \in \mu : x \in S(B)\} \in \mu^\circ\) is false. In fact, let \(X = \{a, b\}\) and \(\mu = \{0_X, A_r : 0 < r \leq \frac{1}{3}\}\), where \(A_r(a) = A_r(b) = r\). Then \((X, \mu)\) is a GFTS. It is easy to see that \(\bigwedge\{B \in \mu : a \in S(B)\} = \bigwedge\{B \in \mu : b \in S(B)\} = 0_X \in \mu\), but \(\min(X, \mu, a) = \min(X, \mu, b) = \phi\). The desired result for \(\min(X, \mu, x)\) corresponding to that in the above theorem goes as follows.

**Theorem 3.5.** Let \((X, \mu)\) be a GFTS and \(x \in X\). Then the following are equivalent:

(a) \(\min(X, \mu, x) \neq \phi\).

(b) \(\bigwedge\{B \in \mu : x \in S(B)\}(x) \neq 0\) and \(\bigwedge\{B \in \mu : x \in S(B)\} \in \mu\).

Proof. (a) \(\Rightarrow\) (b): Suppose \(\min(X, \mu, x) = \{A\}\). Then for each \(B \in \mu\) with \(x \in S(B)\) we have \(A \leq B\). Thus \(A \leq \bigwedge\{B \in \mu : x \in S(B)\}\). Again \(x \in S(A)\) \(\Rightarrow\) \(\bigwedge\{B \in \mu : x \in S(B)\} \leq A\). Thus \(A = \bigwedge\{B \in \mu : x \in S(B)\}\). Hence (b) follows as \(x \in S(A)\) and \(A = \bigwedge\{B \in \mu : x \in S(B)\} \in \mu\).

(b) \(\Rightarrow\) (a): Let \(F = \bigwedge\{B \in \mu : x \in S(B)\}\). Then by (b), \(F \in \mu\). Also by the first condition of (b), \(x \in S(F)\). Now for any \(G \in \mu\) with \(x \in S(G)\), we have \(F \leq G\). Thus \(F \in \min(X, \mu, x)\) and hence \(\min(X, \mu, x) \neq \phi\).

If a fuzzy set \(A\) is locally minimal at some point in a GFTS, then it is not true in general that each point of \(S(A)\) must have a locally minimal \(\mu_f\)-open set. We show this by the following example:

**Example 3.2.** Let \((X, \mu)\) be a GFTS, where \(X = \{a, b\}\), and \(\mu\) consist of \(0_X, 1_X\) and all those fuzzy sets \(P\) such that \(0 < P(a) \leq \frac{1}{2}\) and \(P(b) = 0\). Then \(1_X\) is a locally minimal \(\mu_f\)-open set at \(b\), i.e., \(\min(X, \mu, b) = \{1_X\}\). But \(a \in S(1_X)\), \(\min(X, \mu, a) = \phi\).
We have already defined minimal $\mu f$-closed set. In an analogous way we define minimal $\mu f$-open sets as follows:

**Definition 3.3.** A non-null $\mu f$-open set $U$ in a GFTS $(X, \mu)$ is called a minimal $\mu f$-open set in $X$ if there is no $\mu f$-open set strictly lying between $0_X$ and $U$.

**Theorem 3.6.** Let $(X, \mu)$ be a GFTS, $A \in \mu$ and $p_\lambda$ be any fuzzy point in $X$ such that $p_\lambda \leq A$. Then the following are equivalent:

(a) $A$ is a minimal $\mu f$-open set in $X$, and $\min(X, \mu, p_\lambda) \neq \phi$.

(b) $\min(X, \mu, p_\lambda) = \{A\}$.

Proof. (a)$\Rightarrow$(b): Let $A$ be a minimal $\mu f$-open set in $X$ and $p_\lambda \leq A$. Since $\min(X, \mu, p_\lambda) \neq \phi$, let $B \in \min(X, \mu, p_\lambda)$. Then by definition of $\min(X, \mu, p_\lambda)$, $B \leq A$. Again, $A$ is minimal $\mu f$-open set in $X$ implies $A = B$. Thus $\min(X, \mu, p_\lambda) = \{A\}$.

(b)$\Rightarrow$(a): Let us take $B \in \mu \setminus \{0_X\}$ such that $B \leq A$. Let us choose some $y \in X$ such that $B(y) > 0$. Then $A(y) \geq B(y) > 0$. We consider the fuzzy point $y_\alpha$ where $\alpha = \frac{B(y)}{2}$. Then $y_\alpha \leq A \wedge B$. Now by (b), $\min(X, \mu, y_\alpha) = \{A\} \Rightarrow A \leq B$. So $A = B$ and hence $A$ is a minimal $\mu f$-open set in $X$.

The second condition of (a) is clear from (b).

Remark 2. The implication ‘(a) $\Rightarrow$ (b)’ of the above theorem fails if the second condition of (a) is dropped. In fact, let $X = \{a, b, c\}$, $\mu = \{0_X, A, B, A \vee B\}$, where $A(a) = 0.1$, $A(b) = 0.4$, $A(c) = 0.2$; $B(a) = 0.2$, $B(b) = 0.3$ and $B(c) = 0.5$. Then both $A$ and $B$ are minimal $\mu f$-open sets in the GFTS $(X, \mu)$ and the fuzzy point $a_{0.1} \leq A \wedge B$. But $\min(X, \mu, a_{0.1}) = \phi$ ($\neq \{A\}$ or $\{B\}$).

**Acknowledgement**

The authors are grateful to the referee for his/her meticulous reading of the manuscript and making critical comments, which have gone significantly towards marked improvement of the paper.
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