On Fuzzy V-B-ideals

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Abstract--- In this paper, we present the idea in fuzzy V-B-ideal in V-B-BH algebra, we state and gives some propositions and examples which explain the relationships between these notions and the other types of fuzzy ideal s in BH algebra.

Keyword: V-B-BH algebra, V-B-ideal, fuzzy subst.

INTRODUCTION


On the other hand, we shall mention the development in fuzzy sense. The notions of fuzzy subset presented by L. A. Zadeh In 1965 as a method for representing uncertainty in real physical world [7]. In 2006, C. H. Parkstudied the interval-valued Fuzzy ideal in BH-algebras[1].

In this paper, we present the notions in fuzzy V-B-ideal in V-B-BH-algebra.

I. PRELIMINARIES

In this section, we present some concepts about a B-algebra, such as BCK-algebra, BCI-algebra, BH-algebra and V-B-BH algebra (BC A-part, homomorphism, ideal V-B-ideal) in BH-algebra with some theorems, propositions and examples. Also, we review some fuzzy preliminaries about fuzzy set, level set fuzzy set, fuzzy singleton, fuzzy ideal, (image and preimage) of fuzzy set under function, and some other concepts that we need in our work.

Definition (1.1) :[5]
A B-algebra is a nonempty set $\psi$ with a constant $\varnothing$ and a binary operation "$\odot$" satisfying the following axioms:

i. $\hat{a} \odot \hat{a} = \varnothing$

ii. $\hat{a} \odot \varnothing = \hat{a}$

iii. $(\hat{a} \odot \hat{c}) \odot \hat{g} = \hat{a} \odot (\hat{g} \odot (\varnothing \odot \hat{c}))$,
for every $\hat{a}, \hat{c}, \hat{g} \in \psi$.

Definition (1.2) : [6]
A BCI-algebra is an algebra $(\psi, \odot, \varepsilon)$, where $\psi$ is a nonempty set, "$\odot" is a binary operation and $\varepsilon$ is a constant, satisfying the following axioms:

i. $((\hat{a} \odot \hat{c}) \odot (\hat{a} \odot \hat{g})) \odot (\hat{g} \odot (\varepsilon \odot \hat{c})) = \varepsilon$, for every $\hat{a}, \hat{c}, \hat{g} \in \psi$.

ii. $(\hat{a} \odot (\hat{a} \odot \hat{c})) \odot \hat{c} = \varnothing$, for every $\hat{a}, \hat{c} \in \psi$.

iii. $\hat{a} \odot \varepsilon = \varnothing$, for every $\hat{a} \in \psi$.

iv. $\hat{a} \odot \hat{c} = \varnothing$ and $\hat{c} \odot \hat{a} = \varnothing$

imply $\hat{a} = \hat{c}$, for every $\hat{a}, \hat{c} \in \psi$.

Definition (1.3) : [6]
A BCK-algebra is a BCI-algebra satisfying the axiom: $\varepsilon \odot \hat{a} = \varnothing$, for all $\hat{a} \in \psi$.

Definition (1.4) :[9]
A BH-algebra is a nonempty set $\psi$ with a constant $\varepsilon$ and a binary operation "$\odot$" satisfying the following conditions:

i. $\hat{a} \odot \hat{a} = \varnothing$, for every $\hat{a} \in \psi$.

ii. $\hat{a} \odot \hat{c} = \varepsilon$ and $\hat{c} \odot \hat{a} = \varnothing$

imply $\hat{a} = \hat{c}$, for every $\hat{a}, \hat{c} \in \psi$.

iii. $\hat{a} \odot \varepsilon = \hat{c}$, for every $\hat{a} \in \psi$. 

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Remark (1.5):[8]
Let $X$ and $Y$ be $BH$ – algebras. A mapping $\Omega : X \rightarrow Y$ is called a homomorphism if $\Omega (x \oplus y) = \Omega (x) \oplus \Omega (y)$ for all $x, y \in X$.

A homomorphism $\Omega$ is called a epimorphism if it is injective. The set $\{ x \in X : \Omega (x) = 0' \}$ is called the kernel of $\Omega$, denoted by $Ker(\Omega)$, and the set $\{ \Omega(x) : x \in X \}$ is called image of $\Omega$, denoted by $Im(\Omega)$. Notice that $\Omega(0) = 0'$, for all homomorphism $\Omega$, and $\Omega^{-1}(Y) = \{ x \in X : \Omega (x) = y \text{, for some } y \in Y \}$

Definition (1.6) :[9]
Let $I$ be a nonempty subset of a BH-algebra $\psi$. Then $I$ is said to be ideal of $\psi$ if it satisfies:
i. $0 \in I$.
ii. $\tilde{a} \oplus \tilde{c} \in I$ and $\tilde{c} \in I$ imply $\tilde{a} \in I$.

Definition (1.7): [4]
A $V$-$B$-$BH$-algebra is defined to be a BH-algebra $\psi$ in which there exists a proper subset $V$ of $\psi$, such that:
i. $0 \in V$, $|V| \geq 2$.
ii. $V$ is a $B$-algebra.

Definition (1.8): [4]
A nonempty subset $I$ of a $V$-$B$-algebra $\psi$ is said to be $V$-$B$-ideal of $\psi$ related to $V$ if it satisfies:
i. $0 \in I$.
ii. $\tilde{a} \oplus \tilde{c} \in I$ and $\tilde{c} \in I$ imply $\tilde{a} \in I$.

Definition (1.9) [7]:
Let $\psi$ be a non-empty set. A fuzzy set $\mathcal{A}$ in $\psi$ (a fuzzy subset of $\psi$) is a function from $\psi$ into the real interval $[0,1]$.

Definition (1.10) :[7]
Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy sets in $\psi$, then:
i. $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}(\tilde{a}) \leq \mathcal{B}(\tilde{a})$, $\forall \tilde{a} \in \psi$.
ii. $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}(\tilde{a}) \leq \mathcal{B}(\tilde{a})$, $\forall \tilde{a} \in \psi$.
iii. $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}(\tilde{a}) < \mathcal{B}(\tilde{a})$, for every $\tilde{a} \in \psi$, where $\mathcal{A}$ is called a proper fuzzy subset of $\mathcal{B}$

Definition (1.11) :[2]
Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy sets in $\psi$, then
i. $(\mathcal{A} \cap \mathcal{B})(x) = \min\{ \mathcal{A}(x), \mathcal{B}(x) \}$, for every $x \in \psi$.

ii. $(\mathcal{A} \cup \mathcal{B})(x) = \max\{ \mathcal{A}(x), \mathcal{B}(x) \}$, for every $x \in \psi$.

$\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$

are fuzzy sets in $\psi$.

In general, if $\{ \mathcal{A}_{\epsilon}, \epsilon \in \gamma \}$ is a family of fuzzy sets in $\psi$, then:

$\bigwedge_{\epsilon \in \gamma} \mathcal{A}_{\epsilon}(\tilde{a}) = \inf\{ \mathcal{A}_{\epsilon}(\tilde{a}), \epsilon \in \gamma \}$

for every $\tilde{a} \in \psi$, and

$\bigvee_{\epsilon \in \gamma} \mathcal{A}_{\epsilon}(\tilde{a}) = \sup\{ \mathcal{A}_{\epsilon}(\tilde{a}), \epsilon \in \gamma \}$

for every $\tilde{a} \in \psi$

which are also fuzzy sets in $\psi$.

Definition (1.12) :[7]
Let $\mathcal{A}$ be a fuzzy set in $\psi$. For $t \in [0,1]$, the set $\mathcal{A}_t = \{ \tilde{a} \in \psi : \mathcal{A}(\tilde{a}) \geq t \}$ is called a level subset of $\mathcal{A}$.

Definition (1.13) :[1]
A fuzzy subset $\mathcal{A}$ of a BH-algebra $\psi$ is said to be a fuzzy ideal if and only if:
i. For any $\tilde{a} \in \psi$, $\mathcal{A}(0) \geq \mathcal{A}(\tilde{a})$.
ii. For any $\tilde{a}, \tilde{c} \in \psi$, $\mathcal{A}(\tilde{a}) \geq \min\{ \mathcal{A}(\tilde{a} \oplus \tilde{c}), \mathcal{A}(\tilde{c}) \}$.

Definition (1.14):[3]
Let $\mathcal{A}$ and $\mathcal{B}$ be any two sets, $\mathcal{A} \oplus \mathcal{B}$ be any fuzzy set in $\mathcal{A}$ and $\Omega : \mathcal{A} \rightarrow \mathcal{B}$ be any function. If $\Omega^{-1}(\tilde{c}) = \{ \tilde{a} \in \mathcal{A} : \Omega(\tilde{a}) = \tilde{c} \}$ for $\tilde{c} \in \mathcal{B}$, then the fuzzy set $\mathcal{B} \oplus \mathcal{A}$ is defined by

$\mathcal{B} \oplus \mathcal{B}(\tilde{c}) = \sup\{ \mathcal{A}(\tilde{a}) \in \Omega^{-1}(\tilde{c}) \}$ if $\Omega^{-1}(\tilde{y}) \neq \emptyset$

otherwise
for every $\mathfrak{c} \in B$, is said to be image of $A^\circ$ under $\Omega$ and denoted by $\Omega(A^\circ)$.

**Definition (1.15):**[3]

Let $A$ and $B$ be any two sets, $\Omega: A \rightarrow B$ be any function and $B^\circ$ be any fuzzy set in $\Omega(A)$. The fuzzy set $A^\circ$ in $A$ defined by $A^\circ(\mathfrak{a}) = B^\circ(\Omega(\mathfrak{a}))$ for every $\mathfrak{a} \in \Psi$ is called the preimage of $B^\circ$ under $\Omega$ and is denoted by $\Omega^{-1}(B^\circ)$.

**II. THE MAIN FUZZY RESULTS**

In this section, we present the concept of a fuzzy V-B-ideal of a $V$-B-BH algebra. For our discussion, we relate this idea with other species of fuzzy ideals of a BH-algebra. Also, we stated and proved some theorems and examples about these concepts.

**Definition (2.1):**

A fuzzy subset $A$ of a $V$-B-BH-algebra $\Psi$ is said to be a fuzzy $V - B$ - ideal if and only if:

i. $A(\mathfrak{o}) \geq A(\mathfrak{a})$, for every $\mathfrak{a} \in \Psi$.

ii. $A(\mathfrak{a}) \geq \min\{A(\mathfrak{a} \otimes \mathfrak{c}), A(\mathfrak{c})\}$, for every $\mathfrak{a} \in V, \mathfrak{c} \in \Psi$.

**Example (2.2):**

Consider the $V$-B-BH-algebra $\Psi = \{\mathfrak{o}, 1, 2, 3\}$ with binary operation $\otimes$ defined as follows.

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where $V = \{\mathfrak{o}, 1, 2\}$. The fuzzy subset $A$ of $\Psi$ which is defined by:

$A(\mathfrak{a}) = \begin{cases} 0.5 & \mathfrak{a} = \mathfrak{o} \\ 0.4 & \mathfrak{a} = 1, 2 \\ 0.3 & \mathfrak{a} = 3 \end{cases}$

is a fuzzy $V$-B-ideal of $\Psi$.

**Proposition (2.3):**

Let $\Psi$ be a $V$-B-BH-algebra. Then every fuzzy ideal of $\Psi$ is a fuzzy $V$-B-ideal of $\Psi$.

**Proof:**

Let $A$ be a fuzzy ideal of $\Psi$. To prove that $A$ is a fuzzy $V$-B-ideal of $\Psi$.

$\Rightarrow A(\mathfrak{a}) \geq A(\mathfrak{a})$, for every $\mathfrak{a} \in \Psi$.[Since $A$ is a fuzzy ideal of $\Psi$. By definition (1.13)(i)]

$ii. \Rightarrow A(\mathfrak{a}) \geq \min\{A(\mathfrak{a} \otimes \mathfrak{c}), A(\mathfrak{c})\}$[Since $A$ is a fuzzy ideal. By definition (1.13)(ii)]

Therefore, $A$ is a fuzzy $V$-B-ideal of $\Psi$.

**Remark (2.4):**

The converse of the proposition (2.3) may not be true in general as the following example shows.

**Example (2.5):**

The fuzzy V-B-ideal $A$ of $\Psi$ in example (2.2) is not a fuzzy ideal of $\Psi$, since $A(3) = 0.3 < \min\{A(3 \otimes 1), A(1)\} = A(1) = 0.4$.

**Proposition (2.6):**

Let $\Psi$ be a $V$-B-BH-algebra and let $A$ be a fuzzy $V$-B-ideal of $\Psi$ such that $A(\mathfrak{a} \otimes \mathfrak{c}) \leq \min\{A(\mathfrak{a}), A(\mathfrak{c})\}$ for every $\mathfrak{a} \in V, \mathfrak{c} \in \Psi$. Then $A$ is a fuzzy ideal of $\Psi$.

**Proof:**

Let $A$ be a fuzzy $V - B$ - ideal of $\Psi$.

To prove that $A$ is a fuzzy ideal of $\Psi$.

i. Let $\mathfrak{a} \in \Psi \Rightarrow A(\mathfrak{a}) \geq A(\mathfrak{a})$, for every $\mathfrak{a} \in \Psi$. [By definition (2.1)(i)]

ii. Let $\mathfrak{a}, \mathfrak{c} \in \Psi$.

Then we have two cases.

**Case 1:** $1$. If $\mathfrak{a} \in V \Rightarrow A(\mathfrak{a})$

$\geq \min\{A(\mathfrak{a} \otimes \mathfrak{c}), A(\mathfrak{c})\}$

[By definition (2.1)(ii)]

**Case 2:** $1$. If $\mathfrak{a} \in V \Rightarrow A(\mathfrak{a} \otimes \mathfrak{c})$

$\leq \min\{A(\mathfrak{a}), A(\mathfrak{c})\}$

$\Rightarrow A(\mathfrak{a} \otimes \mathfrak{c}) \leq A(\mathfrak{a})$ and $A(\mathfrak{a} \otimes \mathfrak{c})$

$\leq A(\mathfrak{c}) \Rightarrow A(\mathfrak{a} \otimes \mathfrak{c})$

$= \min\{A(\mathfrak{a} \otimes \mathfrak{c}), A(\mathfrak{c})\}$

$\Rightarrow A(\mathfrak{a}) \geq \min\{A(\mathfrak{a} \otimes \mathfrak{c}), A(\mathfrak{c})\}$

$\Rightarrow A$ is a fuzzy ideal of $\Psi$. 
Theorem (2.7):
Let $\psi$ be a V-B-BH-algebra. Then $A$ is a fuzzy V-B-ideal of $\psi$ if and only if $A_i$ is a V-B-ideal of $\psi$ for every $t \in \{0, \sup A(\tilde{a})\}$.

Proof:
Let $A$ be a fuzzy V-B-ideal of $\psi$ and $t \in \{0, \sup A(\tilde{a})\}$.

To prove that $A_i$ is a V-B-ideal of $\psi$.

i. Since $A(\tilde{a}) \geq A(\tilde{a})$, for every $\tilde{a} \in \psi = A(\tilde{a}) \geq t 
\Rightarrow \sup A(\tilde{a})$.

ii. Let $\tilde{a} \in V$ and $\tilde{c} \in A_i$.

$\Rightarrow A(\tilde{a} \oplus \tilde{c}) \geq t$.

To prove that $A_i$ is a V-B-ideal of $\psi$.

Conversely, $A$ is a fuzzy V-B-ideal of $\psi$.

Proof:

i. Let $t \in \{0, \sup A(\tilde{a})\}$

$\Rightarrow A$ is a V-B-ideal of $\psi$.

$\Rightarrow 0 \in A \Rightarrow A(\tilde{a}) \geq A(\tilde{a})$, for every $\tilde{a} \in \psi$.

[Since $t \in \{0, \sup A(\tilde{a})\}$]

ii. Let $\tilde{a} \in V$, $\tilde{c} \in \psi$ and $t = \min \{A(\tilde{a} \oplus \tilde{c}), A(\tilde{c})\}$.

$\Rightarrow A(\tilde{a} \oplus \tilde{c}) \geq t$.

To prove that $A$ is a fuzzy V-B-ideal of $\psi$.

$A(\tilde{a}) \geq t \Rightarrow A(\tilde{a}) \geq A(\tilde{a})$, for every $\tilde{a} \in \psi$.

$\Rightarrow A$ is a fuzzy V-B-ideal of $\psi$.

Proposition (2.8):
Let $\{A_\epsilon(\tilde{a}) : \epsilon \in \gamma\}$ be a family of fuzzy V-B-ideals of a V-B-BH-algebra $\psi$. Then $\sup_{\epsilon \in \gamma} A_\epsilon(\tilde{a})$ is a fuzzy V-B-ideal of $\psi$.

Proof:
Let $\{A_\epsilon(\tilde{a}) : \epsilon \in \gamma\}$ be a family of fuzzy V-B-ideals of $\psi$.
Theorem (2.10): Let $\mathcal{A}$ be a nonempty subset of an $\mathcal{A}$-ideal of a $\mathcal{A}$-algebra. Let $\mathcal{A}$ be a fuzzy set defined by $\mathcal{A} = \text{sup}\{\mathcal{A} \in \mathcal{A}, \epsilon \in \gamma\}$ and $\mathcal{A} = \text{min}\{\mathcal{A} \in \mathcal{A}, \epsilon \in \gamma\}$. Then $\mathcal{A}$ is a fuzzy $\mathcal{A}$-ideal of $\mathcal{A}$.

Proposition (2.11): Let $\Omega: (\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{A}, \mathcal{B})$ be a $\mathcal{A} \rightarrow \mathcal{B} - \mathcal{B}$-ideal of $\mathcal{A}$, then $\Omega(\mathcal{A})$ is a fuzzy $\mathcal{A}$-ideal of $\mathcal{A}$. 

Proof: Let $\mathcal{A}$ be a fuzzy $\mathcal{A}$-ideal of $\mathcal{A}$.

i. $\mathcal{A}(\mathcal{A}) = \mathcal{A}(\mathcal{A})$

ii. Let $\mathcal{A} \in V$ and $\mathcal{A} \in V$.
Proof :

Let \( \mathcal{A} \) be a fuzzy \( V - B \) - ideals of \( \psi \).

i. Let \( \bar{c} \in \xi \) such that \( \bar{c} = \Omega(\bar{a}) \).

for some \( \bar{a} \in \psi \).

\( (\Omega(\mathcal{A}))(\mathcal{A}(\bar{a})) = \sup\{\mathcal{A}(\bar{a}) \in \Omega^{-1}(\mathcal{A}(\bar{a}))\} \)

\( = \mathcal{A}(\bar{a}) \supseteq \mathcal{A}(\bar{a}) \)

[By definition (2.1)(i)]

\( = (\Omega(\mathcal{A}))((\Omega(\mathcal{A}))(\bar{c})) \)

\( \Rightarrow (\Omega(\mathcal{A}))(\bar{c}) \supseteq (\Omega(\psi))(\bar{c}) \)

i. Let \( \bar{c} \in \Omega(V), \bar{c} \in \xi \),

there exists \( \bar{a}1 \in V \) and \( \bar{a}2 \in \psi \)

such that

\( \bar{c}1 = \Omega(\bar{a}1) \) and \( \bar{c}2 = \Omega(\bar{a}2) \)

\( \Rightarrow (\Omega(\mathcal{A}))(\bar{c}1) = \sup\{\mathcal{A}(\bar{a}) \in \Omega^{-1}(\bar{c}1)\} \)

\( \Omega(\mathcal{A}))(\bar{c}1) \supseteq \mathcal{A}(\bar{a}1) \)

\( \geq \min\{\mathcal{A}(\bar{a}1 \oplus \bar{a}2), \mathcal{A}(\bar{a}2)\} \)

[By definition (2.1)(ii)]

\( = \min\{(\Omega(\mathcal{A}))(\bar{a}1), \Omega(\mathcal{A}))(\bar{a}2)\} \)

\( (\Omega(\mathcal{A}))(\bar{a}1) \) be a fuzzy \( \Omega(V - B) \) - ideal of \( \xi \).

**Theorem (2.12):**

Let \( \Omega: (\psi, \otimes, \Phi) \rightarrow (\xi, \otimes', \Phi) \) be a \( V - B \) - BH - epimorphism, and let \( \mathcal{B} \) be a fuzzy \( V - B \) - ideal of \( \xi \) such that \( \Omega^{-1}(V - B) \) is a \( B \) - algebra. Then \( \Omega^{-1}(\mathcal{B}) \) is a fuzzy \( \Omega^{-1}(V - B) \) - ideal of \( \psi \).

Proof :

i. Let \( \bar{a} \in \psi \). Since \( \Omega(\bar{a}) \in \xi \)

and \( \mathcal{B} \) is a fuzzy \( V - B \) - ideals of \( \xi \),

\( (\Omega^{-1}(\mathcal{B}))(\bar{a}) = \mathcal{B}(\Omega(\bar{a})) \)

\( = \mathcal{B}(\bar{a}) \supseteq \mathcal{B}(\Omega(\bar{a})) \)

\( = (\Omega^{-1}(\mathcal{B}))(\bar{a}) \)

ii. Let \( \bar{a} \in \Omega^{-1}(V), \bar{c} \in \psi \).

\( \Omega^{-1}(\mathcal{B})(\bar{a}) = \mathcal{B}(\Omega(\bar{a})) \)

[By definition (1.15)]

\( \geq \min\{\mathcal{B}(\Omega(\bar{a}) \oplus \bar{c}), \mathcal{B}(\Omega(\bar{c}))\} \)

[By definition (3.1)(ii)]

\( = \min\{\mathcal{B}(\bar{a} \oplus \bar{a}), \mathcal{B}(\bar{c})\} \)

\( = \min\{\Omega^{-1}(\mathcal{B})(\bar{a} \oplus \bar{c}), \Omega^{-1}(\mathcal{B})(\bar{c})\} \)

\( \Rightarrow \Omega^{-1}(\mathcal{B}) \) be a fuzzy \( \Omega^{-1}(V - B) \) - ideal of \( \psi \).

**REFERENCES**


