Characterization of Some Intuitionistic Fuzzy Subsets of Intuitionistic Fuzzy Ideal Topological Spaces

Amneh Kareem Yousif
Amneh.almusawi@yahoo.com
Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar.

Abstract
In this paper we study some subsets of intuitionistic fuzzy ideal topological space like intuitionistic fuzzy regular-I-closed set, intuitionistic fuzzy $A_I$-set, intuitionistic fuzzy $f_I$-set and intuitionistic fuzzy-I-locally closed set. Also, we characterize $\tau$-codense intuitionistic fuzzy ideals in terms of intuitionistic fuzzy $A_I$-set, intuitionistic fuzzy regular-I-closed set and intuitionistic fuzzy-I-locally closed set and prove some result about them.

Keywords: Intuitionistic fuzzy ideal topological spaces, $\tau$-codense intuitionistic fuzzy ideals, Intuitionistic fuzzy $A_I$-set.
وصف بعض المجموعات الضبابية الحدسية الجزئية من الفضاءات
المثالية الضبابية الحدسية
آمنة كريم يوسف
Amneh.almusawi@yahoo.com
قسم الرياضيات كلية التربية للعلوم الصرفة، جامعة ذي قار

الخلاصة

في هذا البحث درسنا بعض المجموعات الجزئية من الفضاء التبليجي المتوازي الضبابي الحدسي مثل المجموعة الضبابية الحدسية المنظمة I المغلقة، المجموعة الضبابية الحدسية، المجموعة المغلقة codense من الفضاءات الضبابية الحدسية، وكذلك قمنا بتميز المثاليات الضبابية الحدسية من ناحية المجموعات الضبابية الحدسية المنظمة I المغلقة، المجموعات الضبابية الحدسيه، المجموعات المغلقة I المثالية المحلية الضبابية الحدسية، المجموعات A، I المفصلة الضبابية الحدسيه، وبرهننا بعض النتائج عليها.
1. Introduction

The concept of intuitionistic fuzzy sets and their operations introduced by Atanassov [1] in 1986 as a generalization of fuzzy sets which published by L.A. Zadeh [8] in 1965. Later many researchers worked on this set as Coker and Saadati [2, 4] who defined the notion of intuitionistic fuzzy topology and studied the basic concepts of intuitionistic fuzzy point [4]. Gain P.K. and et al [3] in 2012 are studied and characterized some fuzzy subsets like fuzzy regular I-closed set, fuzzy $A_I$-set and others and characterized $\tau$-codense fuzzy ideals on fuzzy $A_I$-set, fuzzy regular-I-closed set and fuzzy-I-locally closed set. In this paper, we extended those ideas of fuzzy ideal topological space in intuitionistic fuzzy ideal topological spaces and we generalized some concepts of intuitionistic fuzzy ideal topological space which initiated by Salama [5] and proved some results about them.

2. Preliminaries

Definition (2.1) [1] Let $X$ be a non-empty set. An intuitionistic fuzzy set (IFS) $A$ is an object having the form:

$A=\{(x, \mu_A(x), v_A(x)), x \in X\}$, where the functions $\mu_A : X \rightarrow I$ and $v_A : X \rightarrow I$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set $A$, respectively, and

$0 \leq \mu_A(x) + v_A(x) \leq 1$ for each $x \in X$.

Definition (2.2) [1] $\bar{0} = \{(x, 0,1), x \in X\}$

$\bar{1} = \{(x, 1,0), x \in X\}$
are the intuitionistic fuzzy sets corresponding to empty set and the entire universe respectively.

**Definition (2.3) [1]** Let $X$ be a non-empty set and let $A$ and $B$ are IFSs in the form $A = \{(x, \mu_A(x), v_A(x)), x \in X\}$, $B = \{(x, \mu_B(x), v_B(x)), x \in X\}$.

Then:
1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $v_A(x) \geq v_B(x)$ for all $x \in X$.
2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3) $A^c = \{(x, v_A(x), \mu_A(x)), x \in X\}$.
4) $A \cap B = \{\min\{\mu_A(x), \mu_B(x)\}, \max\{v_A(x), v_B(x)\}, x \in X\}$.
5) $A \cup B = \{\max\{\mu_A(x), \mu_B(x)\}, \min\{v_A(x), v_B(x)\}, x \in X\}$.

**Definition (2.4) [1]** Let $\{A_i, i \in J\}$ be an arbitrary family of IFSs in $X$, then

1) $\cap_i A_i = \{(x, \land_i \mu_A_i(x), \lor_i v_A_i(x)), x \in X\}$
2) $\cup_i A_i = \{(x, \lor_i \mu_A_i(x), \land_i v_A_i(x)), x \in X\}$.

**Definition (2.5) [2, 4]** An intuitionistic fuzzy topology (IFT for short) on a nonempty set $X$ is a family $\tau$ of an intuitionistic fuzzy sets in $X$ satisfying the following conditions:

1) $0, 1 \in \tau$
2) For any $G_1, G_2 \in \tau$, then $G_1 \cap G_2 \in \tau$
3) For any family $\{G_i, i \in J\} \subseteq \tau$, then $\cup_i G_i \in \tau$.

The pair $(X, \tau)$ is called intuitionistic fuzzy topological space (IFTS for short), and any intuitionistic fuzzy set in $\tau$ is known as an intuitionistic fuzzy open set (IFOS for short) in $X$.

**Definition (2.6) [2]** The complement $A^c$ of an intuitionistic fuzzy open set in $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS for short) in $X$.

**Definition (2.7) [2]** Let $(X, \tau)$ is an intuitionistic fuzzy topological space and $A = \{(x, \mu_A(x), v_A(x)), x \in X\}$ be an intuitionistic fuzzy set in $X$. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of $A$ are defined as follows:

$cl(A) = \bar{A} = \cap \{K: K \text{ is an IFCS in } X \text{ and } A \subseteq K\}$

$int(A) = A^o = \cap \{G: G \text{ is an IFOS in } X \text{ and } G \subseteq A\}$

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Definition (2.8) [2] Let $X$ be a nonempty set, an intuitionistic fuzzy point denoted by $\chi_{(\alpha, \beta)}$ is an intuitionistic fuzzy set $A = \{(x, \mu_A(x), v_A(x)), x \in X\}$ such that

\[ \mu_A(y) = \begin{cases} \alpha & \text{if } y = x \\ e.w & \text{otherwise} \end{cases} \quad \text{and} \quad v_A(y) = \begin{cases} \beta & \text{if } y = x \\ e.w & \text{otherwise} \end{cases} \]

where $x \in X$ is a fixed point, and constants $\alpha, \beta \in I$, satisfy $\alpha + \beta \leq 1$.

Definition (2.9) [5] Let $X$ be a nonempty set and $L$ a nonempty family of intuitionistic fuzzy sets. We call $L$ is an intuitionistic fuzzy ideal (IFL for short ) on $X$ if:

1) $A \in I$ and $B \leq A \implies B \in L$.
2) $A, B \in L \implies A \lor B \in L$.

Let $(X, \tau)$ is an intuitionistic fuzzy topological space and $L$ is an intuitionistic fuzzy ideal, then $(X, \tau, I)$ is said to be an intuitionistic fuzzy ideal topological space.

Definition (2.10) [5] Let $(X, \tau)$ is an intuitionistic fuzzy topological space and $L$ be is an intuitionistic fuzzy ideal on $X$, then the intuitionistic fuzzy local function $A^*(L, \tau)$ of $A$ is the union of all intuitionistic fuzzy points (IFP for short) $\chi_{(\alpha, \beta)}$ such that if $U \in N_{\chi_{(\alpha, \beta)}}$ and

\[ A^*(L, \tau) = \bigvee \{ \chi_{(\alpha, \beta)} \in \text{IFP}: A \land U \not\in L \text{ for any } U \in N_{\chi_{(\alpha, \beta)}} \} \]

$A^*(L, \tau)$ is called an intuitionistic fuzzy local function of $A$ with respect to $\tau$ and $L$ which it will be denoted by $A^*(L, \tau)$ or simply $A^*(L)$.

Theorem (2.11) [5] Let $(X, \tau)$ is an intuitionistic fuzzy topological space and $L$ be an intuitionistic fuzzy ideal on $X$. Then for any intuitionistic fuzzy sets $A, B$ of $X$, the following statements are satisfied:

1) $A \subseteq B \implies A^*(L, \tau) \subseteq B^*(L, \tau)$.
2) $A^* = cl(A^*) \subseteq cl(A)$.
3) $A^{**} \subseteq A^*$.
4) $(A \lor B)^* = A^* \lor B^*$.
5) $(A \land B)^* \subseteq A^* \land B^*$.

Lemma (2.12) Let $(X, \tau, I)$ is an intuitionistic fuzzy ideal topological space and $U \in \tau$, then $U \land A^* \subseteq (U \land A)^*$. 

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**Definition (2.13)**[6, 7] An intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \nu_A(x)), x \in X\} \) in an intuitionistic fuzzy topological space \((X, \tau)\) is said to be an:

1) Intuitionistic fuzzy semiopen set (IFSOS in short) if \( A \subseteq \text{cl}(\text{int}(A)) \).
2) Intuitionistic fuzzy preopen set (IFPOS in short) if \( A \subseteq \text{int}(\text{cl}(A)) \).
3) Intuitionistic fuzzy \( \alpha \) –open set (IF\( \alpha \)OS in short) if \( A \subseteq \text{int}(\text{cl}(\text{int}(A))) \).
4) Intuitionistic fuzzy \( \beta \) open set (IF\( \beta \)OS in short) if \( A \subseteq \text{cl}(\text{int}(\text{cl}(A))) \).
5) Intuitionistic fuzzy regular closed set (IFRCS in short) if \( A = \text{cl}(\text{int}(A)) \).
6) Intuitionistic fuzzy regular open set (IFROS in short) if \( A = \text{int}(\text{cl}(A)) \).

**Definition (2.14)** An intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \nu_A(x)), x \in X\} \) in an intuitionistic fuzzy topological space \((X, \tau)\) is said to be an:

1) Intuitionistic fuzzy locally closed set if \( A = U \land V \), where \( U \in \tau \) and \( V \) is an intuitionistic fuzzy closed set (IFLC(X)).
2) Intuitionistic fuzzy A-set if \( A = U \land V \), where \( U \in \tau \) and \( V \) is an intuitionistic fuzzy regular closed set (IFA(X)).

**Definition (2.15)** An intuitionistic fuzzy set \( A = \{(x, \mu_A(x), \nu_A(x)), x \in X\} \) in an intuitionistic fuzzy ideal topological space \((X, \tau, I)\) is said to be an:

1) Intuitionistic fuzzy \( I \) –open if \( A \subseteq \text{int}(A) \) (IFIO for short).
2) Intuitionistic fuzzy \( * \) –dense – in – itself if \( A \subseteq A^* \).
3) Intuitionistic fuzzy \( \tau^* \) – closed if \( A^* \subseteq A \).
4) Intuitionistic fuzzy \( * \) – perfect if \( A = A^* \).
5) Intuitionistic fuzzy \( \alpha - I \) – open if \( A \subseteq \text{int}(\text{cl}^*(\text{int}(A))) \) (IF\( \alpha \)IO in short).
6) Intuitionistic fuzzy semi-\( I \)-open if \( A \subseteq \text{cl}^*(\text{int}(A)) \) (IFSIOS in short).
7) Intuitionistic fuzzy pre-\( I \)-open if \( A \subseteq \text{int}(\text{cl}^*(A)) \) (IFPIOS in short).

**Definition (2.16)** In an intuitionistic fuzzy ideal topological space \((X, \tau, I)\), \( I \) is said to be \( \tau \) – codense if \( I \land \tau = \{0\} \).
Lemma(2.17) For an intuitionistic fuzzy ideal topological space \((X, \tau, I)\), \(I\) is said to be \(\tau\) - codense if and only if \(\bar{I} = \bar{I}^*\).

Lemma(2.18) For an intuitionistic fuzzy ideal topological space \((X, \tau, I)\), then the following are equivalent:

1) \(I\) is \(\tau\) - codense
2) \(\bar{I} = \bar{I}^*\).
3) For every \(A \in \tau\), \(A \subseteq A^*\).
4) For every \(A \in IFSO(X)\), \(A \subseteq A^*\).
5) For every intuitionistic fuzzy regular closed set \(C\), \(C = C^*\).

Proof: 1 \(\iff\) 2 follows from lemma(2.17)

2 \(\implies\) 3

Let \(A \in \tau\), since \(\bar{I} = \bar{I}^*\) by using lemma(2.17)

\[\Rightarrow I \land \tau = \{\overline{0}\}\]
\[\Rightarrow A \notin I\]
\[\Rightarrow A \land U \notin I \quad \forall U \in N_{X(a,\beta)}\]
\[\Rightarrow A \in A^*\]

Thus, \(A \subseteq A^*\).

3 \(\implies\) 4

Let \(A \in IFSO(X)\)

\[\Rightarrow\] there exists an intuitionistic fuzzy set \(H\) in \(X\) such that

\[H \subseteq A \subseteq cl(H)\).

For any intuitionistic fuzzy subset \(H\) in \(X\), by Theorem(2.11)

We have \(H^* = cl(H^*) \subseteq cl(H)\).

Since \(H\) is intuitionistic fuzzy open, \(H \subseteq H^* \implies cl(H) \subseteq cl(H^*)\)
and so \(H^* = cl(H^*) = cl(H)\).

Therefore, \(A \subseteq cl(H) = cl(H^*) = H^* \subseteq A^*\).

\[\Rightarrow A \subseteq A^*\]
4 \Rightarrow 5 \quad \text{Let } C \text{ is an intuitionistic fuzzy regular closed, then } C \text{ is an intuitionistic fuzzy semiopen and intuitionistic fuzzy closed.}

Since \( C \in IFSO(X) \Rightarrow C \subseteq C^* \)
Since C is an intuitionistic fuzzy closed
\[ C = cl^*(C) = C \lor C^* \Rightarrow C^* \subseteq C. \]
Hence, \( C^* = C. \)

5 \Rightarrow 1

Since \( \bar{1} \) is an intuitionistic fuzzy regular closed
\[ \Rightarrow \text{by hypothesis } \bar{1} = \bar{1}^* \]
\[ \Rightarrow \text{by lemma(2.17) we get } I \text{ is } \tau - \text{codense.} \]

**Lemma(2.19):** Let \( (X, \tau, I) \) be an intuitionistic fuzzy ideal topological space. If an intuitionistic fuzzy subset \( A \) of \( X \) is an intuitionistic fuzzy \( * - \) dense in itself, then \( A^* = cl(A) = cl^*(A) \).

**Proof:** Since \( A \) is \( * - \) dense in itself, \( A \subseteq A^* \).

By Theorem(2.11) for any intuitionistic fuzzy subset \( A \) of \( X \),
\[ A^* = cl(A^*) \subseteq cl(A) \quad (1) \]
Since \( A \subseteq A^* \Rightarrow cl(A) \subseteq cl(A^*) \quad (2) \)
From (1) and (2) we get \( cl(A) = cl(A^*) \)
\[ \Rightarrow A^* = cl(A^*) = cl(A). \]
Also, \( cl(A) = cl^*(A) \)
Hence, \( A^* = cl(A) = cl^*(A) \).

**Lemma(2.20):** Let \( (X, \tau, I) \) be an intuitionistic fuzzy ideal topological space and let
\[ \Delta = \{ A : A \text{ is an intuitionistic fuzzy subset of } X \text{ and } A \subseteq A^* \} . \]

Then, \( \Delta \land I = \{ \bar{0} \} . \)

**Proof:** Let \( A \in \Delta \land I \Rightarrow A \in \Delta \) and \( A \in I \)
\[ A \in I \Rightarrow A^* = \bar{0} \]
\[ A \in \Delta \Rightarrow A \subseteq A^* = \bar{0} \]
Therefore, \( A = \bar{0} . \)
Hence, \( \Delta \land I = \{ \bar{0} \} . \)

**Definition (3.1):** An intuitionistic fuzzy subset $A$ of an intuitionistic fuzzy ideal topological space $(X, \tau, I)$ is said to be intuitionistic fuzzy regular $I$-closed if $A = (\text{int}(A))^*$.

**Definition (3.2):** An intuitionistic fuzzy subset $A$ of an intuitionistic fuzzy ideal topological space $(X, \tau, I)$ is said to be

1) Intuitionistic fuzzy $A_I$-set if $A = U \land V$ where $U \in \tau$ and $V$ is an intuitionistic fuzzy regular $I$-closed set (IF$A_I(X)$).

2) Intuitionistic fuzzy $I$-locally closed set if $A = U \land V$ where $U \in \tau$ and $V$ is an intuitionistic fuzzy $* -$perfect set (IFILC(X)).

**Lemma (3.3):** Let $(X, \tau, I)$ be an intuitionistic fuzzy ideal topological space. An intuitionistic fuzzy subset $A$ of $X$ is an intuitionistic fuzzy $I$-locally closed if and if $A = U \land A^*$ for some $U \in \tau$.

**Proof:** Let $A$ is an IFILC set in $(X, \tau, L)$

\[ \Rightarrow A = U \land V, U \in \tau \quad \text{and} \quad V \text{ is an intuitionistic fuzzy } \ast -$perfect set \]

\[ \Rightarrow V = V^* \]

Since $A \subseteq V$

By Theorem (2.11) $A^* \subseteq V^*$.

Also, by Lemma (2.12) $A^* = (U \land V)^* \supseteq U \land V^* = U \land V = A$

\[ \Rightarrow A \subseteq A^* \]

\[ \Rightarrow A = A \land A^* = (U \land V) \land A^* = U \land (V \land A^*) = U \land A^* \]

\[ \Rightarrow A = U \land A^*. \]

**Theorem (3.4):** Every intuitionistic fuzzy $A_I$-set in $(X, \tau, I)$ is an intuitionistic fuzzy $I$-locally closed.

**Proof:** Let $A$ is an intuitionistic fuzzy $A_I$-set in $(X, \tau, I)$.

\[ \Rightarrow A = U \land V \text{ where } U \in \tau \quad \text{and} \quad V \text{ is an intuitionistic fuzzy regular } I \text{-closed.} \]

\[ \Rightarrow V = (\text{int}(V))^* \]

Since $\text{int}(V) \subseteq V \Rightarrow (\text{int}(V))^* \subseteq V^*$
\[ V = (\text{int}(V))^* \subseteq V^* \Rightarrow V \subseteq V^*(1) \]

Since \( V = (\text{int}(V))^* \Rightarrow V^* = ((\text{int}(V))^*)^* \subseteq (\text{int}(V))^* = V \)
\[ \Rightarrow V^* \subseteq V(2) \]

From (1) and (2) we get \( V = V^* \)

Therefore, \( V \) is an intuitionistic fuzzy \( * - \) perfect set.

Hence \( A \) is an intuitionistic fuzzy \( I \)-locally closed set.

**Theorem (3.5):** Let \( A \) is an intuitionistic fuzzy open set in \((X, \tau, I)\). \( A \) is an intuitionistic fuzzy \( A_I \)-set if and only if \( A \) is an intuitionistic fuzzy \( I \)-locally closed set.

**Proof:**

\((\Rightarrow)\) Let \( A \) intuitionistic fuzzy \( A_I \)-set in \((X, \tau, I)\).

\[ \Rightarrow \text{by Theorem (3.4) } A \text{ is an intuitionistic fuzzy } I \text{-locally closed set.} \]

\((\Leftarrow)\) Let \( A \) is an intuitionistic fuzzy \( I \)-locally closed set and fuzzy open set.

\[ \Rightarrow A = U \land A^* \text{ for some } U \in \tau \]

\[ \Rightarrow A \subseteq A^* \]

Since \( \text{int}(A^*) \subseteq A^* \)

\[ \Rightarrow ((\text{int}(A))^*)^* \subseteq (A^*)^* \subseteq A^* \Rightarrow ((\text{int}(A))^*)^* \subseteq A^*(1) \]

Since \( A \subseteq A^* \) and \( A \) is an intuitionistic fuzzy open set

\[ \Rightarrow A^* = (\text{int}(A))^* \subseteq ((\text{int}(A))^*)^* \]

\[ \Rightarrow A^* \subseteq ((\text{int}(A))^*)^* \text{ (2)} \]

From (1) and (2) we get \( A^* = ((\text{int}(A))^*)^* \)

\[ \Rightarrow A^* \text{ is an intuitionistic fuzzy regular } I \text{-closed set.} \]

Hence \( A \) is an intuitionistic fuzzy \( A_I \)-set.

**Theorem (3.6):** Let \((X, \tau)\) is an intuitionistic fuzzy topological space and \( A \) is an intuitionistic fuzzy subset of \( X \) then \( A \) is an intuitionistic fuzzy \( A \)-set if and only if \( A \) is an intuitionistic fuzzy semiopen and intuitionistic fuzzy locally closed set.

**Proof:** Let \( A \) is an intuitionistic fuzzy \( A \)-set in \((X, \tau)\)

\[ \Rightarrow A = U \land F \text{ where } U \in \tau \text{ and } F \text{ is an intuitionistic fuzzy regular closed} \]

\[ \Rightarrow F \text{ is an intuitionistic fuzzy closed.} \]

\[ \Rightarrow F \text{ is an intuitionistic fuzzy locally closed.} \]
Since \( A = U \land F \Rightarrow \text{int}(A) = \text{int}(U \land F) \)
\[ \Rightarrow \text{int}(A) = U \land \text{int}(F). \]
So, \( A = U \land \text{cl}(\text{int}(F)) \subseteq \text{cl}(U \land \text{int}(F)) = \text{cl}(\text{int}(A)) \)
\[ A \subseteq \text{cl}(\text{int}(A)). \]
Therefore, \( A \) is an intuitionistic fuzzy semiopen.
Conversely, let \( A \) is an intuitionistic fuzzy semiopen and intuitionistic fuzzy locally closed set.
\[ \Rightarrow A \subseteq \text{cl}(\text{int}(A)) \text{and } A = U \land \text{cl}(A), U \in \tau \]
\[ \Rightarrow \text{cl}(A) \subseteq \text{cl}(\text{cl}(\text{int}(A))) = \text{cl}(A) \subseteq \text{cl}(\text{int}(A)) \](1)
Since, \( \text{int}(A) \subseteq A \Rightarrow \text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \)(2)
From (1) and (2) we get \( \text{cl}(A) = \text{cl}(\text{int}(A)). \)
Therefore, \( \text{cl}(A) \) is an intuitionistic fuzzy regular closed.
Hence, \( A \) is an intuitionistic fuzzy A-set.

**Definition (3.7):** An intuitionistic fuzzy subset \( A \) of IFITS (\( X, \tau, I \)) is said to be intuitionistic fuzzy \( f_I \)-set if \( A \subseteq \left( \text{int}(A) \right)^* \).
The family of all intuitionistic fuzzy \( f_I \)-set in \( X \) denoted by \( \text{IF}^f_I(X) \).

**Theorem (3.8):** If \( A \) is an intuitionistic fuzzy \( A_I \)-set in IFLTS (\( X, \tau, I \)) then the following holds:

1) \( A \) and \( \text{int}(A) \) are *-dense-in-itself.

2) \( A^* = \text{cl}(A) = \text{cl}^*(A) \text{and} \left( \text{int}(A) \right)^* = \text{cl}(\text{int}(A)). \)

3) \( A \) is an intuitionistic fuzzy fuzzy \( f_I \)-set.

4) \( A^* = \left( \text{int}(A) \right)^* = \left( \left( \text{int}(A) \right)^* \right)^* = (A^*)^* \).

5) \( A^* \text{and} \left( \text{int}(A) \right)^* \) are intuitionistic fuzzy *-perfect and intuitionistic fuzzy I-locally closed sets.

6) \( A^* \) is an intuitionistic fuzzy regular-I-closed.

**Proof:** 1) Let \( A \) is an intuitionistic fuzzy \( A_I \)-set in IFLTS (\( X, \tau, I \))
\[ \Rightarrow A = U \land V, \text{where} U \in \tau \text{ and } V \text{ is an intuitionistic fuzzy regular-I-closed set.} \]
\[ \Rightarrow V = (\text{int}(V))^* \]

\[ \Rightarrow A = U \land V = U \land (\text{int}(V))^* \subseteq (U \land \text{int}(V))^* = (\text{int}(A))^* \subseteq A^* \]

\[ \Rightarrow A \subseteq A^* \]

Since \( \text{int}(A) \subseteq A \subseteq (\text{int}(A))^* \subseteq A^* \)

\[ \Rightarrow \text{int}(A) \subseteq (\text{int}(A))^* \text{ and } A \subseteq A^* \]

Therefore, \( A \) and \( \text{int}(A) \) are intuitionistic fuzzy \(^*\)-dense-in-itself.

2) by Lemma (2.4) \( A^* = \text{cl}(A) = cl^*(A) \)

and \( (\text{int}(A))^* = cl(\text{int}(A)). \)

3) From (1), \( A \subseteq (\text{int}(A))^* \)

So, \( A \) is an intuitionistic fuzzy \( f_I \)-set.

4) From (1), \( \text{int}(A) \subseteq A \subseteq (\text{int}(A))^* \subseteq A^* \)

And so \((\text{int}(A))^* \subseteq A^* \subseteq ((\text{int}(A))^*)^* \subseteq (\text{int}(A))^* \subseteq A^* \)

Therefore, \( A^* = (\text{int}(A))^* = ((\text{int}(A))^*)^* = (A^*)^* \).

5) From (4) \( A^* = (A^*)^* \text{ and } (\text{int}(A))^* = ((\text{int}(A))^*)^* \)

Then, \( A^* \text{ and } (\text{int}(A))^* \) are intuitionistic fuzzy \(^*\)-perfect and hence are intuitionistic fuzzy I-locally closed sets.

6) From (4), \( A^* = (\text{int}(A))^* \)

Let \( B = (\text{int}(A))^* \)

\[ \Rightarrow (\text{int}(B))^* = (\text{int}(\text{int}(A))^*)^* = (\text{int}(A^*))^* \supseteq (\text{int}(A))^* = B \]

\[ \Rightarrow B \subseteq (\text{int}(B))^* \quad (1) \]

Since \( \text{int}(B) \subseteq B \Rightarrow \text{int}(B)^* \subseteq B^* = ((\text{int}(A))^*)^* \subseteq (\text{int}(A))^* \subseteq B \)

\[ \Rightarrow (\text{int}(B))^* \subseteq B^* \quad (2) \]

From (1) and (2), we obtain \((\text{int}(B))^* = B\)

Then \( B \) is an intuitionistic fuzzy regular I-locally closed set and so \( A^* \) is an intuitionistic fuzzy regular I-locally closed set.

**Theorem (3.9)** In any IFLTS \((X, \tau, I), IF\!A_1(X) \land I = \{0\}.\)
Proof: Let $A \in IFA_I(X) \land I \Rightarrow A \in IFA_I(X)$. By Theorem(3.8), $A$ is an intuitionistic fuzzy *-dense-in-itself

$\Rightarrow A \subseteq A^*$

By lemma(2.20), $IFA_I(X) \land I = \{\emptyset\}$.

**Theorem(3.10)** An intuitionistic fuzzy subset $A$ of an IFLTS $(X, \tau, I)$ is an intuitionistic fuzzy $A_I$-set if and if $A$ is both intuitionistic fuzzy $f_I$-set and intuitionistic fuzzy $I$-locally closed set.

**Proof:** Let $A$ is an $IFA_I$-set

By Theorem(3.4), $A$ is an intuitionistic fuzzy $I$-locally closed set.

Also, $A = U \land V$ where $U \in \tau$ and $V$ is an intuitionistic fuzzy regular-$I$-closed set

$\Rightarrow V = (\text{int}(V))^*$

$\Rightarrow \text{int}(A) = \text{int}(U \land V) = U \land (\text{int}(V)) \Rightarrow (\text{int}(A))^* = (U \land \text{int}(V))^*$

Since $A = U \land V$

$\Rightarrow A = U \land (\text{int}(V))^* \subseteq (U \land \text{int}(V))^* = (\text{int}(A))^*$

$\Rightarrow A \subseteq (\text{int}(A))^*$

Hence, $A$ is an intuitionistic fuzzy $f_I$-set.

Conversely, let $A$ is both intuitionistic fuzzy $f_I$-set and intuitionistic fuzzy $I$-locally closed set.

$\Rightarrow A \subseteq (\text{int}(A))^* \Rightarrow A^* \subseteq ((\text{int}(A))^*)^* \subseteq (\text{int}(A))^* \subseteq A^*$

$\Rightarrow A^* = (\text{int}(A))^*$

By Theorem(3.8), $A^*$ is an intuitionistic fuzzy regular $-I -$ closed set.

Since $A$ is an intuitionistic fuzzy $I$-locally closed set.

By lemma(3.3), $A = U \land A^*$ for some $U \in \tau$.

Since $A^*$ is an intuitionistic fuzzy regular $-I -$ closed set, Then $A$ is an intuitionistic fuzzy $A_I$-set.

**Theorem(3.11)** Let $(X, \tau, I)$ be an intuitionistic fuzzy ideal topological space and let $A \in \tau$, then $A$ is intuitionistic fuzzy $f_I$-set if and only if $A$ is an intuitionistic fuzzy $A_I$-set.
Proof: let $A \in \tau$ and $A$ is an intuitionistic fuzzy $f_i$-set

$\Rightarrow A \subseteq (\text{int}(A))^* \subseteq A^* \text{ and } A^* = (\text{int}(A))^*$

By Theorem (3.8), $A^*$ is an intuitionistic fuzzy regular $\neg I$-closed set.

Since $A = A \land A^*$

$\Rightarrow A$ is an intuitionistic fuzzy $A_i$-set.

Conversely, let $A$ is an intuitionistic fuzzy $A_i$-set.

By Theorem (3.10), $A$ is an intuitionistic fuzzy $f_i$-set.

Theorem (3.12) In any intuitionistic fuzzy ideal topological space $(X, \tau, I)$, $IF_{f_i}(X) \land I = \{\bar{0}\}$.

Proof: Let $A \in IF_{f_i}(X) \land I$

$\Rightarrow A \in IF_{f_i}(X) \text{ and } A \in I$.

$A \in I \Rightarrow A^* = \bar{0}$.

$A \in IF_{f_i}(X) \Rightarrow A \subseteq (\text{int}(A))^* \subseteq A^* = \bar{0} \Rightarrow A = \bar{0}$

$\Rightarrow IF_{f_i}(X) \land I = \{\bar{0}\}$.

Theorem (3.13) Let $(X, \tau, I)$ be an intuitionistic fuzzy ideal topological space and $A$ is an intuitionistic fuzzy subset of $X$. Then $A$ is an intuitionistic fuzzy regular $\neg I$-closed set if and only if $A$ is both intuitionistic fuzzy $f_i$-set and intuitionistic fuzzy $\tau^*$-closed set.

Proof: Let $A$ is an intuitionistic fuzzy regular $\neg I$-closed in $(X, \tau, I)$.

$\Rightarrow A = (\text{int}(A))^* \Rightarrow A \subseteq (\text{int}(A))^* \Rightarrow A$ is an intuitionistic fuzzy $f_i$-set.

Since $\text{int}(A) \subseteq A \Rightarrow (\text{int}(A))^* \subseteq A^*$

$\Rightarrow A = (\text{int}(A))^* \subseteq A^* \Rightarrow A \subseteq A^*$

$\Rightarrow A^* = ((\text{int}(A))^*)^* \subseteq (\text{int}(A))^* = A \Rightarrow A^* \subseteq A$

$\Rightarrow A^* = A$

Then $A$ is an intuitionistic fuzzy $\tau^*$-closed set.

Conversely, let $A$ is both intuitionistic fuzzy $f_i$-set and intuitionistic fuzzy $\tau^*$-closed set.

$\Rightarrow A \subseteq (\text{int}(A))^*$ and $A^* \subseteq A$. 
Since \( \text{int}(A) \subseteq A \Rightarrow (\text{int}(A))^* \subseteq A^* \)
\[ \Rightarrow (\text{int}(A))^* \subseteq A^* \subseteq A \subseteq (\text{int}(A))^* \]
Therefore, \( A = (\text{int}(A))^* \)
Hence \( A \) is an intuitionistic fuzzy regular-I-closed set.

**Theorem (3.14)** Every intuitionistic fuzzy \( f_I \)-set of an intuitionistic fuzzy ideal topological space \( (X, \tau, I) \) is an intuitionistic fuzzy semi-I-open set.

**Proof:** Let \( A \) is an intuitionistic fuzzy \( f_I \)-set.
\[ \Rightarrow A \subseteq (\text{int}(A))^* \subseteq \text{cl}^*(\text{int}(A)) \]
\[ \Rightarrow A \subseteq \text{cl}^*(\text{int}(A)). \]
Then \( A \) is an intuitionistic fuzzy semi-I-open set.

**Theorem (3.15)** For an intuitionistic fuzzy ideal topological space \( (X, \tau, I) \), \( I \) is \( \tau \)-codense if and if \( A \in \tau \), then \( A \in IFA_I(X) \).

**Proof:** Let \( I \) is \( \tau \)-codense and \( A \in \tau \)
By lemma(2.18), for any \( A \in \tau \), \( A \subseteq A^* \Rightarrow A = A \wedge A^* \).
Since \( A \in \tau \Rightarrow (\text{int}(A))^* = A^* \)
By Theorem(3.8), \( A^* \) is an intuitionistic fuzzy regular-I-closed set.
\[ \Rightarrow \text{a is an intuitionistic fuzzy} \ A_I \text{-set}. \]
Conversely, if we have \( A \in \tau \Rightarrow A \in IFA_I(X) \)
By Theorem(3.9), \( IFA_I(X) \wedge I = \{0\} \Rightarrow I \wedge \tau = \{0\} \).
Therefore, \( I \) is \( \tau \)-codense.

**Corollary (3.16)** Let \( (X, \tau, I) \) be an intuitionistic fuzzy ideal topological space. Then the following statements are equivalent:
1) \( I \) is \( \tau \)-codense.
2) \( \tau = \text{IFPIO}(X) \wedge IFA_I(X) \).
3) \( \tau = \text{IF}_a(X) \wedge IFA_I(X) \).
4) \( A \in \tau \Rightarrow A \in IFA_I(X) \).

**Proof:** obvious.
Theorem (3.17) Let \((X, \tau, I)\) be an intuitionistic fuzzy ideal topological space. Then \(I\) is \(\tau\)-codense if and only if \(IFRIC(X) = IFRC(X)\).

Proof: Let \(I\) is \(\tau\)-codense.

By lemma (2.18), \(A \in \tau, A \subseteq A^* \Rightarrow A\) is an intuitionistic fuzzy \(*\)-dense-in-itself.

Let \(A \in IFRIC(X) \iff A = (int(A))^*\)

\(\iff (int(A))^* = cl(int(A))\) by lemma (2.19)

\(\Rightarrow A = cl(int(A)) \iff A \in IFRC(X)\).

Conversely, let \(IFRIC(X) = IFRC(X)\)

Since \(\bar{1}\) is an IFRC \(\Rightarrow \bar{1}\) is an IFRIC

\(\bar{1} = (int(\bar{1}))^* = \bar{1}^* \Rightarrow\) by lemma (2.18), \(I\) is \(\tau\)-codense.

Theorem (3.18) Let \((X, \tau, I)\) be an intuitionistic fuzzy ideal topological space. Then \(I\) is \(\tau\)-codense if and only if \(IF_A I (X) = IFA(X)\).

Proof: Let \(I\) is \(\tau\)-codense.

Let \(A \in IF_A I (X) \Rightarrow A = U \land V\) where \(A \in \tau\) and \(V\) is an IFRIC set

\(\Rightarrow V = (int(V))^*\)

\(\Rightarrow cl(V) = cl((int(V))^*) = (int(V))^* = V\) by Theorem (2.11).

Also, by Theorem (2.11), \((int(V))^* \subseteq cl(int(V))\)

\(\Rightarrow V = (int(V))^* \subseteq cl(int(V)) \subseteq cl(V) = V\)

\(\Rightarrow V = cl(int(V))\)

\(\Rightarrow V\) is an IFRC(X).

\(A \in IFA(X)\).

Let \(B \in IFA(X)\)

\(\Rightarrow B = S \land T\) where \(S \in \tau\) and \(T \in IFRC(X)\)

\(\Rightarrow\) by Theorem (3.17), \(T \in IFRIC(X)\)

\(\Rightarrow B \in IF_A I (X)\).

Conversely, let \(IF_A I (X) = IFA(X)\).

Since \(\bar{1}\) is an IFA-set \(\Rightarrow \bar{1}\) is an IFA\(_I\) set.
by Theorem(3.4) \Rightarrow \overline{I} \subseteq \overline{I}^* \Rightarrow \overline{I} \text{ is } *\text{-dense-in-itself.}

Then by lemma(2.18), I is \tau -\text{codense.}

\textbf{Theorem(3.19)} For an intuitionistic fuzzy ideal topological space \((X, \tau, I)\), \(IFILC(X) \wedge I = \{\overline{0}\}.

\textbf{Proof:} \(A \in IFILC(X).

\Rightarrow A = U \wedge V \text{ where } A \in \tau \text{ and } V \text{ is an IF } *\text{-perfect set}

\Rightarrow V = V^*.

\text{Since } A \subseteq V \Rightarrow A^* \subseteq V^*

\Rightarrow A^* = (U \wedge V)^* \supseteq U \wedge V^* = U \wedge V = A

\Rightarrow A \subseteq A^*

\Rightarrow \text{by lemma(2.20), } IFILC(X) \wedge I = \{\overline{0}\}.

\textbf{Theorem(3.20)} Let \((X, \tau, I)\) be an intuitionistic fuzzy ideal topological space. Then the following statements are equivalent:

1) I is \tau -\text{codense.}

2) \(\tau = IFPIO(X) \wedge IFILC(X).

3) \(\tau = IF_{\alpha}IO(X) \wedge IFILC(X).

4) A \in \tau \Rightarrow A \in IFILC(X).

\textbf{Proof:} \(1 \Rightarrow 2\), Let I is \(\tau -\text{codense.}

\text{From corollary(3.16), } \tau = IFPIO(X) \wedge IFA_{\ell}(X).

\text{By Theorem(3.5), } \tau = IFPIO(X) \wedge IFILC(X).

2 \Rightarrow 3, \tau = IFPIO(X) \wedge IFILC(X).

\text{Let } A \in \tau \Rightarrow A \in IFPIC(X) \Rightarrow A \subseteq int(cl^*(A)).

\text{Since } A \in \tau \Rightarrow A = int(A)

\Rightarrow A \subseteq int(cl^*(int(A)))

\Rightarrow A \in IF_{\alpha}IO(X)

\Rightarrow \tau = IF_{\alpha}IO(X) \wedge IFILC(X)

3 \Rightarrow 4 \text{ and } 2 \Rightarrow 4 \text{ are obvious.}
4 \Rightarrow 1 \text{let } A \in \tau \Rightarrow A \in \text{IFILC}(X).

By Theorem(3.19), \text{IFILC}(X) \wedge I = \{\emptyset\} \Rightarrow \tau \wedge I = \{\emptyset\}.

Therefore, \(I\) is \(\tau\)-codense.

1 \Rightarrow 3 \text{let } I \text{ is } \tau \text{-codense and } A \in \tau

By lemma(2.18), \(A \in \tau \Rightarrow A \subseteq A^*\)

Since \(\text{int}(A) = A \Rightarrow \text{cl}(\text{int}(A)) = \text{cl}(A)\)

\(\Rightarrow \text{cl}^*(\text{int}(A)) = \text{cl}^*(A)\)

\(\Rightarrow \text{int}\left(\text{cl}^*(\text{int}(A))\right) = \text{int}(\text{cl}^*(A))\)

Since \(A \subseteq \text{cl}(A) \Rightarrow A^* \subseteq \text{cl}^*(A) \Rightarrow A \subseteq A^* \subseteq \text{cl}^*(A)\)

\(\Rightarrow \text{int}(A) \subseteq \text{int}(A^*) \subseteq \text{int}(\text{cl}^*(A)) = \text{int}\left(\text{cl}^*(\text{int}(A))\right)\).

\(\Rightarrow A \subseteq \text{int}\left(\text{cl}^*(\text{int}(A))\right)\)

Hence \(A \in \text{IF}_a\text{IO}(X)\).

Since \(A \subseteq A^* \Rightarrow A = A \wedge A^*\)

\(A \subseteq A^* \Rightarrow A^* \subseteq (A^*)^* \subseteq A^* \Rightarrow (A^*)^* = A^*\)

\(\Rightarrow A^*\) is an IF *-perfect set.

\(\Rightarrow A \in \text{IFILC}(X)\).

Conversely, let \(A \in \text{IFILC}(X)\) and \(A \in \text{IF}_a\text{IO}(X)\).

\(A \in \text{IFILC}(X) \Rightarrow A = U \wedge A^*\) for some \(U \in \tau\).

\(A \in \text{IF}_a\text{IO}(X) \Rightarrow A \subseteq \text{int}\left(\text{cl}^*(\text{int}(A))\right) \subseteq \text{int}\left(\text{cl}^*(A)\right) = \text{int}\left(\text{cl}^*(U \wedge A^*)\right) \subseteq \text{int}(\text{cl}^*(A^*))\).

\(\Rightarrow A \subseteq \text{int}(\text{cl}^*(A^*)) = \text{int}(A^*)\)

\(\Rightarrow A \subseteq \text{int}(A^*)\)

Since \(A \subseteq U \Rightarrow A \subseteq U \wedge \text{int}(A^*) = \text{int}(U \wedge A^*) = \text{int}(A)\)

\(\Rightarrow A \subseteq \text{int}(A)\)

Since \(\text{int}(A) \subseteq A\)

Then \(\text{int}(A) = A\) and therefore \(A \in \tau\).
References


