* – Connectedness in Intuitionistic Fuzzy Ideal Bitopological spaces

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Abstract:

In This paper we introduce the nation of *– Connectedness in Intuitionistic Fuzzy Ideal Bitopological Space . we obtain several properties of *– Connectedness in Intuitionistic Fuzzy Ideal Bitopological spaces and the relationship between this notion and other related notions.

Keywords:

Intuitionistic Fuzzy Ideal Bitopological Spaces ,
Pairwise *– Connected intuitionistic fuzzy sets ,
Pairwise *– Separated intuitionistic fuzzy sets ,
Pairwise *– Connected intuitionistic fuzzy Ideal Bitopological Space

Kelly introduced the concept of "bitopological space" as extension of topological space [4] in 1963.
Mohammed (2015) introduced the notion of "intuitionistic fuzzy ideal bitopological space" [9].

The purpose of this paper is to introduce and study the notion of "\(*\) – connectedness in intuitionistic fuzzy ideal bitopological space". We study the notion of "pairwise \(*\) – connected intuitionistic fuzzy ideal bitopological space".

2. Preliminaries:

Definition 2.1. [7]:-
Let X be a non–empty set and I = [0, 1] be the closed interval of the real numbers. A fuzzy subset \( \mu \) of X is defined to be membership function \( \mu : X \rightarrow I \), such that \( \mu (x) \in I \) for every \( x \in X \). The set of all fuzzy subsets of X denoted by \( I^X \).

Definition 2.2 [5]:-
An intuitionistic fuzzy set (IFs, for short) A is an object have the form:
\[ A = \{ x, \mu_A(x), \nu_A(x) > ; x \in X \} \], where the functions \( \mu_A : X \rightarrow I \), \( \nu_A : X \rightarrow I \) denote the degree of membership and the degree of non–membership of each element \( x \in X \) to the set A respectively, and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \), for each \( x \in X \). The set of all intuitionistic fuzzy sets in X denoted by IFS (x).

Definition 2.3. [3]:-
\( 0_\sim = (< x, 0, 1 >) \), \( 1_\sim = (< x, 1, 0 >) \) are the intuitionistic sets corresponding to empty set and the entire universe respectively.
Definition 2.4. [2] :-

Let $X$ be a non-empty set. An intuitionistic fuzzy point (IFP, for short) denoted by $x_{(\alpha, \beta)}$ is an intuitionistic fuzzy set have the form

$$x_{(\alpha, \beta)}(y) = \begin{cases} < x, \alpha, \beta > & ; \ x = y \\ < x, 0,1 > & ; \ x \neq y \end{cases},$$

where $x \in X$ is a fixed point, and $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$. The set of all IFPs denoted by IFP$(x)$. If $\in$ IFs$(x)$. We say the $x_{(\alpha, \beta)} \in \ A$ if and only if $\alpha \leq \mu_A(x)$ and $\beta \geq \nu_A(x)$, for each $x \in X$.

Definition 2.5. [2] :-

Let $\mu_A(x), \nu_A(x) \mu_B(x), \nu_B(x)$ be two intuitionistic fuzzy sets in $X$. $A$ is said to be quasi-coincident with $B$ (written $AqB$) if and only if, there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$, otherwise $A$ is not quasi-coincident with $B$ and denoted by $\tilde{A}qB$.

Definition 2.6. [2] :-

Let $x_{(\alpha, \beta)} \in$ IFP$(X)$ and $\in$ IFs$(X)$. We say that $x_{(\alpha, \beta)}$ quasi-coincident with $A$ denoted $x_{(\alpha, \beta)}qA$ if and only if, $\alpha > \nu_A(x)$ or $\beta < \mu_A(x)$, otherwise $x_{(\alpha, \beta)}$ is not quasi-coincident with $A$ and denoted by $x_{(\alpha, \beta)}\tilde{q}A$.

Definition 2.7. [2] :-

Let $x_{(\alpha, \beta)}$ be an intuitionistic fuzzy point in $X$ and $A = \{< x, \mu_A(x), \nu_A(x) >, \ x \in X \}$ an IFS in $X$. Suppose further $\alpha$ and $\beta$ are real numbers between 0 and 1. The intuitionistic fuzzy point $x_{(\alpha, \beta)}$ is said to be properly contained in $A$ if and only if, $\alpha < \mu_A(x)$ and $\beta > \nu_A(x)$.
Definition 2.8.[2] :-
An intuitionistic fuzzy point \( x_{(\alpha, \beta)} \) is said to be belong to an intuitionistic fuzzy set \( A \) in \( X \), denoted by \( x_{(\alpha, \beta)} \in A \) if \( \alpha \leq \mu_A(x) \) and \( \beta \geq \nu_A(x) \).

Proposition 2.9. [3] :-
Let \( A, B \) be IFSs and \( x (\alpha, \beta) \) an IFP in \( X \). Then
1- \( A \sqsupseteq B \iff A \subseteq B \)
2- \( A \sqsubseteq B \iff A \subseteq B^c \),
3- \( x_{(\alpha, \beta)} \in A \iff x_{(\alpha, \beta)} \in A^c \),
4- \( x_{(\alpha, \beta)} \notin A \iff x_{(\alpha, \beta)} \notin A^c \).

Proposition 2.10. [8] :-
For \( A, B \in \text{IFS} \) and \( x_{(\alpha, \beta)} \in \text{IFP} (X) \), we have:
\[ A \sqsubseteq B \ \text{if and only if} \ \text{for} \ x_{(\alpha, \beta)} \in A \ \text{then} \ x_{(\alpha, \beta)} \in B \]
\[ \text{ii} - A \sqsubseteq B \ \text{if and only if} \ \text{for} \ x_{(\alpha, \beta)} \notin A \ \text{then} \ x_{(\alpha, \beta)} \notin B \]

Lemma 2.11. [10] :-
Let \( A, B \) and \( C \) be intuitionistic fuzzy sets. If \( q(A \cup B) \), then \( C \sqsupseteq A \) or \( C \sqsupseteq B \).

Definition 2.12. [3] :-
An intuitionistic fuzzy topology (IFT, for short) on a non empty set \( X \) is a family \( \tau \) of an intuitionistic fuzzy set in \( X \) such that
\[ (i) \ 0_\infty, 1_\infty \in \tau , \]
\[ (ii) \ G_1 \cap G_2 \in \tau \ , \text{for any} \ G_1, G_2 \in \tau , \]
\[ (iii) \ \cup \ G_i \in \tau \ , \text{for any arbitrary family} \ \{ G_i : i \in I \} \subseteq \tau . \]
In this case the pair \( (X, \tau) \) is called an intuitionistic fuzzy topological space (IFTS, in short).
**Definition 2.13. [3] :-**

Let \((X, \tau)\) be an intuitionistic fuzzy topological space and 
\[ A = \{<x, \mu_A(x), \nu_A(x)>, \ x \in X\} \]
be an intuitionistic fuzzy set in \(X\) then , an intuitionistic fuzzy interior and intuitionistic fuzzy closure of \(A\) are respectively defined by
\[ \text{int} (A) = A^\circ = \bigcup \{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\} \]
\[ \text{cl} (A) = \overline{A} = \bigcap \{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\}. \]


A non–empty collection of intuitionistic fuzzy sets \(L\) of a set \(X\) is called intuitionistic fuzzy ideal on \(X\) (IFI, for short) such that :

1. If \(A \in L\) and \(B \subseteq A \Rightarrow B \in L\) (heredity)
2. If \(A \in L\) and \(B \in L \Rightarrow A \cup B \in L\) (finite additivity). If \((X, \tau)\) be an IFTS , then the triple \((X, \tau, L)\) is called an intuitionistic fuzzy ideal topological space (IFITS, for short).

**Definition 2.15. [1] :-**

Let \((X, \tau, L)\) be an IFITS. If \(\in \text{ IFS}(X)\). Then the intuitionistic fuzzy local function \(A^*(L, \tau)\) (\(A^*\), for short) of \(A\) in \((X, \tau, L)\) is the union of all intuitionistic fuzzy points \(x_{(\alpha, \beta)}\) such that:

\[ A^*(L, \tau) = \bigvee \{x_{(\alpha, \beta)} : A \cap U \notin L\}, \text{ for every } e \in \text{ IFS}(x_{(\alpha, \beta)}, \tau) \}, \text{ where } \]
\[ \text{N}(x_{(\alpha, \beta)}, \tau) \] is the set of all quasi–neighborhoods of an IFP \(x_{(\alpha, \beta)}\) in \(\tau\).

The intuitionistic fuzzy closure operator of an IFS \(A\) is defined by
\[ \text{cl}^*(A) = A \cup A^*\], and \(\tau^*(L)\) is an IFT finer than \(\tau\) generated \(\text{cl}^*(\cdot)\) and defined as
\[ \tau^*(L) = \{A : \text{cl}^*(A^C) = A^C\}. \]
Lemma 2.16. [8] :-
Let \((X, \tau, L)\) be an IFITS and \(B \subseteq A \subseteq X\). Then
\[
B^*(\tau_A, L_A) = B^*(\tau, L) \cap A.
\]

Lemma 2.17. [8] :-
Let \((X, \tau, L)\) be an IFITS and \(B \subseteq A \subseteq X\). Then
\[
\text{cl}^*_{A}(B) = \text{cl}^*(B) \cap A.
\]

Definition 2.18. [8] :-
An intuitionistic fuzzy set (IFS) \(A\) of intuitionistic fuzzy ideal topological space \((X, \tau, L)\) is said to be *-dense if \(\text{cl}^*(A) = X\).

An intuitionistic fuzzy ideal topological space \((X, \tau, L)\) is said to be *-hyperconnected if IFS \(A\) is *-dense for every IF open subset \(A \neq \emptyset\) of \(X\).

Lemma 2.19. [8] :-
Let \((X, \tau, L)\) be an IFITS for each \(v \in \tau^*, \tau^*_v = (\tau_v)^*\).

Lemma 2.20. [8] :-
Let \((X, \tau, L)\) be an IFITS, \(A \subseteq Y \subseteq X\) and \(Y \in \tau\). The following are equivalent
(1) \(A\) is *-IF open in \(Y\), (2) \(A\) is *-IF open in \(X\).

Proof :- (1) \(\Rightarrow\) (2) let \(A\) be *-IF open in \(Y\). Since \(Y \in \tau \subseteq \tau^*\), by lemma (2.19), \(A\) is *-IF open in \(X\).

Let \(A\) be *-IF open in \(X\). By lemma (2.19), \(A = A \cap Y\) is *-IF open in \(X\). (2) \(\Rightarrow\) (1).

Definition 2.21. [8] :-
Two non empty intuitionistic fuzzy sets \(A\) and \(B\) of an intuitionistic fuzzy ideal topological space \((X, \tau, L)\) are said to be intuitionistic fuzzy...
* separated sets ( IF * separated sets, for short ) if cl*(A) ∩ B and A ∩ cl(B).

**Definition 2.22. [8] :-**

An intuitionistic fuzzy set E in intuitionistic fuzzy ideal topological space (X, τ, L) is said to be intuitionistic fuzzy * connected if it cannot be expressed as the Union of two intuitionistic fuzzy * separated sets. Otherwise, E is said to be intuitionistic fuzzy * disconnected.

If = X, then X is said to be intuitionistic fuzzy * connected space.

**Definition 2.23. [8] :-**

Let τ₁ and τ₂ be two intuitionistic fuzzy topologies on a non-empty set X. The Triple (X, τ₁, τ₂) is called an intuitionistic fuzzy bitopological space (IFBTS, for short), every member of τᵢ is called τᵢ intuitionistic fuzzy open set (τᵢ IFOS), i ∈ {1, 2} and the complement of τᵢ IFOS is τᵢ intuitionistic fuzzy closed set (τᵢ IFCS), i ∈ {1, 2}.

**Example 2.24.[8] :-**

Let X = {e, d} and A, B ∈ IFS(X) such that

X =< x, (0.3, 0.1), (0.5, 0.6) >,
B =< x, (0.2, 0.4), (0.7, 0.3) >. Let τ₁ = {0₁, 1₁, A} and τ₂ = {0₂, 1₂, B} be two IFTS on X. Then (X, τ₁, τ₂) is IFBTS.

**Definition 2.25.[8] :-**

Let (X, τ₁, τ₂) be an IFBTS, A ∈ IFS(X) and x_(α, β) ∈ IFP(X). Then A is said to be quasi-neighborhood of x_(α, β) if there exists a τᵢ IFOS B, i ∈ {1, 2} such that x_(α, β) ∩ B ⊆ A. The set of all quasi –
Definition 2.26. [8] :

An intuitionistic fuzzy bitopological space \((X, \tau_1, \tau_2)\) with an intuitionistic fuzzy ideal \(L\) on \(X\) is called intuitionistic fuzzy ideal bitopological space \((X, \tau_1, \tau_2, L)\) and denoted by IFLBTS

Example 2.27. [8] :

Let \(X = \{e\}\) and \(A, B \in IFS(X)\) such that \(= \langle X, 0.3, 0.5 \rangle\), \(B = \langle X, 0.2, 0.4 \rangle\). Let \((X, \tau_1, \tau_2)\) be an IFLBTS, where \(\tau_1 = \{0_-, 1_-, A\}\) and \(\tau_2 = \{0_-, 1_-, B\}\). If \(L = \{0_-, A, C : C \in IFS(X)\) and \(C \leq A\}\) be an IFL on \(X\). Then \((X, \tau_1, \tau_2)\) is IFLBTS.

Definition 2.28. [8] :

Let \((X, \tau_1, \tau_2, L)\) be an IFLBTS and \(\in IFS(X)\). Then the intuitionistic fuzzy local function of \(A\) in \((x, \tau_1, \tau_2, L)\) denoted by \(A^*(L, \tau_i)\), \(i \in \{1, 2\}\) and defined by as follows:

\[A^*(L, \tau_i) = \bigvee \{x(\alpha, \beta) : A \land U \notin L, \text{ for every } \in N(x(\alpha, \beta), \tau_i)\}, i \in \{1, 2\}\]

Definition 2.29. [8] :

Let \((X, \tau_1, \tau_2)\) be an IFBTS and \(\in IFS(X)\). Then intuitionistic fuzzy interior and intuitionistic fuzzy cloure of \(A\) with respect to \(\tau_i, i \in \{1, 2\}\) are defined by:

\[\tau_i \ominus \text{int } (A) = \bigvee \{G : G \text{ is a } \tau_i \ominus \text{IFOS}, G \leq A\}\]

\[\tau_i \ominus \text{cl } (A) = \bigwedge \{K : K \text{ is a } \tau_i \ominus \text{IFCS}, A \leq K\}\]

Proposition 2.30. [8] :

Let \((X, \tau_1, \tau_2)\) be an IFBTS and \(\in IFS(X)\). Then we have:

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(i) $\tau_i - \text{int}(A) \leq A, i \in \{1, 2\}$

(ii) $\tau_i - \text{int}(A)$ is a largest $\tau_i - \text{IFOS}$ contains in $A$

(iii) $A$ is a $\tau_i - \text{IFOS}$ if and only if $\tau_i - \text{int}(A) = A$

(iv) $\tau_i - \text{int}(\tau_i - \text{int}(A)) = \tau_i - \text{int}(A)$.

(v) $A \leq \tau_i - \text{cl}(A), i \in \{1, 2\}$.

(vi) $\tau_i - \text{cl}(A)$ is smallest $\tau_i - \text{IFCS}$ contains $A$.

(vii) $A$ is a $\tau_i - \text{IFCS}$ if and only if $\tau_i - \text{cl}(A) = A$.

(viii) $\tau_i - \text{cl}(\tau_i - \text{cl}(A)) = \tau_i - \text{cl}(A)$

(ix) $[\tau_i - \text{int}(A)]^c = \tau_i = \text{cl}(A^c), i \in \{1, 2\}$.

(x) $[\tau_i - \text{cl}(A)]^c = \tau_i = \text{int}(A^c), i \in \{1, 2\}$.

Definition 2.31. [8] :-

We define $* -$ intuitionistic fuzzy closure operator for intuitionistic fuzzy bitopology $\tau_i^*(L)$ as follows:

$\tau_i - \text{cl}^*(A) = A \vee A^*(L, \tau_i)$ for every $A \in \tau_i - \text{IFS}(X)$. Also, $\tau_i^*(L)$ is called an intuitionistic fuzzy bitopology generated by $\tau_i - \text{cl}^*(A)$ and defined as:

$\tau_i^*(L) = \{A : \tau_i - \text{cl}^*(A^c) = A^c, i \in \{1, 2\}\}$.

Note: $\tau_i^*(L)$ finer than intuitionistic fuzzy bitopology $\tau_i$ , ( i . e $\tau_i \leq \tau_i^*(L)$).

Remark 2.32. [8] :-

(i) If $L = \{0_\sim\} \Rightarrow A^*(L, \tau_i) = \tau_i - \text{cl}(A)$, for any $A \in \text{IFS}(X)$

$\Rightarrow \tau_i - \text{cl}^*(A) = A \vee A^*(L, \tau_i) = A \vee \tau_i - \text{cl}(A) = \tau_i - \text{cl}(A)$

$\Rightarrow \tau_i^*(\{0_\sim\}) = \tau_i, i \in \{1, 2\}$.

(ii) If $L = \text{IFS}(X) \Rightarrow A^*(L, \tau_i) = 0_\sim$, for any $A \in \text{IFS}(X)$

$\Rightarrow \tau_i - \text{cl}^*(A) = A \vee A^*(L, \tau_i) = A \vee 0_\sim = A$

$\Rightarrow \tau_i^*(L)$ is the intuitionistic fuzzy discrete bitopology on $X$.
3. Main Results

3.1 * − Connectedness in Intuitionistic fuzzy Ideal Bitopological Spaces

Definition 3.1.1 :-

Two non empty $\tau_i$ − intuitionistic fuzzy sets $A$ and $B$ of an intuitionistic fuzzy ideal bitopological space $(X, \tau_1, \tau_2, L)$, $i \in \{1,2\}$, are said to be intuitionistic fuzzy $*-$ separated sets $(\tau_i \text{− IF } *-$ separated sets , for short ) , $i \in \{1,2\}$ if

$\tau_1 - cl^* (A) \bar{q} B$ and $A \bar{q} \tau_1 - cl (B)$

Proposition 3.1.2 :-

Let $A$ and $B$ be an $\tau_i$ − intuitionistic fuzzy $*-$separated sets in IFLBT $(X, \tau_1, \tau_2, L)$, $A, B$ are two non empty $\tau_i$ − intuitionistic fuzzy $*-$separated sets such that $A_1 \leq A$ and $B_1 \leq B$ then $A_1$ and $B_1$ are $\tau_i$ − intuitionistic fuzzy $*-$separated sets in $X$, $i \in \{1,2\}$.

Proof :-

Since $A_1 \leq A$ and $B_1 \leq B$, we have

$\tau_i - cl^* (A_1) \leq \tau_i - cl^* (A)$ and $\tau_i - cl(B_1) \leq \tau_i - cl(B)$, Since $A, B$ are $\tau_i$ − intuitionistic fuzzy $*-$separated then ,

$\tau_i - cl^*(A) \bar{q} B$ and $\bar{q} \tau_i - cl (B)$, $i \in \{1,2\}$

Therefore $\tau_i - cl^*(A) \bar{q} B$ we get $\tau_i - cl^*(A_1) \bar{q} B_1$

And $\bar{q} \tau_i - cl (B)$, and also we get $A_1 \bar{q} \tau_i - cl(B_1)$, $i \in \{1,2\}$

Then $A_1$ and $B_1$ are $\tau_i$ − IF $*-$separated.

Theorem 3.1.3 :-

Let $A$ be $\tau_i$ − intuitionistic fuzzy open set $(\tau_i$ − IFOS) , $i \in \{1,2\}$ and $B$ be $*-$ $\tau_i$ − intuitionistic fuzzy open set in intuitionistic fuzzy ideal
Proof:

(⇒) suppose that \( A \cap B \), then exists an element \( x \in X \) such that \( \mu_A(x) > v_B(x) \) or \( v_A(x) < \mu_B(x) \), and since \( A \subseteq \tau_i - \text{cl}^*(A) \) and \( \subseteq \tau_i - \text{cl}(B) \), \( i = \{1,2\} \)
This implies \( \mu_{\tau_i-\text{cl}^*(A)}(x) > v_B(x) \) or \( v_{\tau_i-\text{cl}^*(A)}(x) < \mu_B(x) \)
And \( \mu_A(x) > v_{\tau_i-\text{cl}(B)}(x) \) or \( v_A(x) < \mu_{\tau_i-\text{cl}(B)}(x) \), \( i \in \{1,2\} \)
Then \( \tau_i - \text{cl}^*(A) \cap B \) and \( A \cap \tau_i - \text{cl}(B) \), \( i \in \{1,2\} \)
This is contradiction. Hence \( A \cap B \).

(⇐) Suppose that \( A \cap B \).
By proposition (2.9), we have \( A \subseteq B^c \)
Since \( B^c \) is \( \tau_i - \text{intuitionistic fuzzy closed set} \), \( i \in \{1,2\} \)
Therefore, \( \tau_i - \text{cl}^*(A) \leq \tau_i - \text{cl}^*(B^c) = B^c \), \( i \in \{1,2\} \) → \( \tau_i - \text{cl}^*(A) \leq B^c \)
Hence by proposition (2.9), we get \( \tau_i - \text{cl}^*(A) \cap (B^c)^c \).
Then \( \tau_i - \text{cl}^*(A) \cap \cdot \) → (1)
Let \( \leq B^c \), since \( B^c \) is \( \ast - \text{IFCS in X} \).
Therefore, \( \tau_i - \text{cl}(A) \leq \tau_i - \text{cl}(B^c) = B^c \), \( i \in \{1,2\} \)
Hence by proposition (2.9), we have \( \tau_i - \text{cl}(A) \cap (B^c)^c \), then \( \tau_i - \text{cl}(A) \cap \cdot \)
Since \( A \subseteq \tau_i - \text{cl}(A) \) and \( \subseteq \tau_i - \text{cl}(B) \), \( i \in \{1,2\} \)
Thus \( A \cap \tau_i - \text{cl}(B) \)...(2)
From (1) and (2) we get \( A \) and \( B \) are \( \tau_i - \text{IF} \ast - \text{separated sets in X} \).
Proposition 3.1.4 :-

Let $A$ be an $\ast -\tau_i -\text{IFCS}$ and $B$ is an $\tau_i -\text{IFCS}$ , $i \in \{1,2\}$ in intuitionistic fuzzy ideal bitopological space $(X, \tau_1, \tau_2, L)$.

Then $A$ and $B$ are $\tau_i -\text{IF} \ast -\text{Separted sets in X if and only if } qB$.

Proof :-

$(\Rightarrow)$ suppose that $A, B$ are $\tau_i -\text{IF} \ast -\text{separated sets in X}$.

$\Rightarrow \tau_i - cl^*(A)qB \text{ and } q\tau_i - cl(B)$, $i \in \{1,2\}$

Since $A$ is $\ast -\tau_i -\text{IFCS}$, then $\tau_i - cl^*(A) = A$, $i \in \{1,2\}$, we get $AqB$

$(\Leftarrow)$ Suppose that $AqB$

Since $A$ is $\ast -\tau_i -\text{IFCS}$ and $B$ is $\tau_i -\text{IFCS}$, $i \in \{1,2\}$

Therefore , $\tau_i - cl^*(A) = A$ and $\tau_i - cl(B) = B$, $i \in \{1,2\}$

We get $\tau_i - cl^*(A)qB$ and $Aq\tau_i - cl(B)$

Hence $A, B$ are $\tau_i -\text{IF} \ast -\text{separated sets in X}$.

Definition 3.1.5 :-

An $\tau_i -\text{intuitionistic fuzzy set (}\tau_i -\text{IFS}) A$ of intuitionistic fuzzy ideal bitopological space $(X, \tau_1, \tau_2, L)$ is said to be $\ast -\tau_i -\text{dense}$ if $\tau_i - cl^*(A) = X$, $i \in \{1,2\}$

An IF ideal bitopological space $(X, \tau_1, \tau_2, L)$ is said to be $\ast -\text{hyperconnected}$ if $\tau_i -\text{IFS} A$ is $\ast -\tau_i -\text{dense}$ for every $\tau_i -\text{IF open subset } A \neq \emptyset$ of $X$, $i \in \{1,2\}$.

Theorem 3.1.6 :-

Let $(X, \tau_1, \tau_2, L)$ be an intuitionistic fuzzy ideal bitopological space and $A,B$ are $\tau_i -\text{intuitionistic fuzzy sets}$ such that $A, B \subset Y \subset X$. Then $A$ and $B$ are $\tau_i -\text{IF} \ast -\text{separated in Y if and only if } A, B$ are $\tau_i -\text{IF} \ast -\text{separated in X}$.

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Proof :- It follows from lemma (2.17) that $\tau_i - \text{cl}^*(A)\bar{q}B$ and $A\bar{q}\tau_i - \text{cl}(B)$, $i \in \{1,2\}$.

**Proposition 3.1.7 :-**

Let $A$ be an $\tau_i$-intuitionistic fuzzy open set ($\tau_i$-IFOS) and $B$ is an $* - \tau_i$-intuitionistic fuzzy open set ($* - \tau_i$-IFOS) in IFLBTS $(X, \tau_1, \tau_2, L)$. Then the sets $C_A B = A \land B^c$ and $C_B A = B \land A^c$ are $\tau_i$-IF $* -$ separated in $X$.

Proof :-

Since $C_A B = A \land B^c$, $C_A B \leq B^c$

$\tau_i - \text{cl}^*(C_A B) \leq \tau_i - \text{cl}^*(B^c) = B^c$ because $B^c$ is $* - \tau_i$-IFCS.

By proposition (2.9) we get

$\tau_i - \text{cl}^*(C_A B)\bar{q}(B^c) \Rightarrow \tau_i - \text{cl}^*(C_A B)\bar{q}B$, $i \in \{1,2\}$

Since $C_B A \leq B$

Therefore $\tau_i - \text{cl}^*(C_A B)\bar{q}C_B A$ ... (1)

$C_B A \leq A^c$

$\tau_i - \text{cl}(C_B A) \leq \tau_i - \text{cl}(A^c) = A^c$, $i \in \{1,2\}$

$\tau_i - \text{cl}(C_B A) \leq A^c$

$\Rightarrow \tau_i - \text{cl}(C_B A)\bar{q}(A^c) \Rightarrow \tau_i - \text{cl}(C_B A)\bar{q}A$, $i \in \{1,2\}$

Since $C_A B \leq A$

Then $\tau_i - \text{cl}(C_B A)\bar{q}C_A B$ ... (2)

From (1) and (2) we get, $C_A B$, $C_B A$ are $\tau_i$-IF $* -$ separated set in $X$.

**Proposition 3.1.8 :-**

Let $A$ be an $* - \tau_i$-intuitionistic fuzzy closed set ($* - \tau_i$-IFCS) and $B$ be $\tau_i$-intuitionistic fuzzy closed set ($\tau_i$-IFCS) in IFLBTS...
\((X, \tau_1, \tau_2, L)\). Then the \(\tau_i -\text{IFS}\) \(C_\bar{A}B = A \land B^c\) and \(C_BA = B \land A^c\) are \(\tau_i -\text{IF } \ast -\text{separated sets in } X, i \in \{1,2\}\).

**Proof :**

Since \(A\) is \(\ast -\tau_i -\text{IFCS}\) and \(B\) is an \(\tau_i -\text{IFCS}\), \(i \in \{1,2\}\)

So \(A = \tau_i - \text{cl}^*\(A\)\) and \(B = \tau_i - \text{cl}(B)\)

\(C_\bar{A}B \leq A \Rightarrow \tau_i - \text{cl}^*(C_\bar{A}B) \leq \tau_i - \text{cl}^*(A) = A, i \in \{1,2\}\)

By proposition ( 2.9 ) we get

\(\tau_i - \text{cl}^*(C_\bar{A}B)\bar{q}A^c\)

Since \(C_BA \leq A^c\), then \(\tau_i - \text{cl}^*(C_\bar{A}B)\bar{q}C_BB \ldots (1)\)

Since \(C_BA \leq B \Rightarrow \tau_i - \text{cl}(C_BA) \leq \tau_i - \text{cl}(B) = B, i \in \{1,2\}\)

By proposition ( 2.9 ) we get

\(\tau_i - \text{cl}(C_BA)\bar{q}B^c\)

Since \(C_\bar{A}B \leq B^c\), then \(\tau_i - \text{cl}(C_BA)\bar{q}C_\bar{A}B \ldots (2)\)

\(C_\bar{A}B, C_BA\) are \(\tau_i -\text{IF } \ast -\text{separated sets in } X\).

**Theorem 3.1.9 :-**

Let \((X, \tau_1, \tau_2, L)\) be IFLBTS. Then \(A\) and \(B\) are two \(\tau_i -\text{IF } \ast -\text{separated sets if and only if there exists an } \tau_i -\text{intuitionistic fuzzy open set } (\tau_i -\text{IFOS})U\) and \(\ast -\tau_i -\text{intuitionistic fuzzy open set } V\) \((\ast \tau_i -\text{IFOS})\), \(i \in \{1,2\}\)

Such that \(A \leq U, B \leq V, A\hat{q}V\) and \(B\hat{q}U\).

**Proof :**

\((\Rightarrow)\) Suppose that \(A, B\) are \(\tau_i -\text{IF } \ast -\text{separated sets}\).

\(\Rightarrow \tau_i - \text{cl}^*(A)\hat{q}B\) and \(A\hat{q}\tau_i - \text{cl}(B)\)

Now put \(V = (\tau_i - \text{cl}^*(A))^c\) and \(U = (\tau_i - \text{cl}(B))^c\)

So \(U\) is \(\tau_i -\text{IFOS}\) and \(V \ast -\tau_i -\text{IFOS}, i \in \{1,2\}\)

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Then $V^c \tilde{q} B$ and $A \tilde{q} U^c$

By proposition (2.9) we get $V^c \leq B^c \Rightarrow B \leq V$ and $A \leq U$

So $A \leq (\tau_i - \text{cl}(B))^c$ and $B \leq (\tau_i - \text{cl}^*(A))^c$

Since $B \leq \tau_i - \text{cl}(B)$ and since $\tau_i - \text{cl}^*(A) = A \vee A^*(L, \tau_i)$, $i \in \{1,2\}$, then $A \leq \tau_i - \text{cl}^*(A)$

Then $A \leq V^c$ and $B \leq U^c$

Therefore, $A \tilde{q} V$ and $B \tilde{q} U$.

($\iff$) Suppose that there exist $U$ be $\tau_i - \text{IFos}$ and $V$ be $* - \tau_i - \text{IFOS}$ in $X$ such that $A \leq U$, $B \leq V$, $A \tilde{q} V$ and $B \tilde{q} U$.

Now $U^c$ is $\tau_i - \text{IFCS}$ and $V^c$ is an $* - \tau_i - \text{IFCs}$ in $X$, $i \in \{1,2\}$

Since $A \tilde{q} V$ and $B \tilde{q} U$, then $A \leq V^c$ and $B \leq U^c$.

Since $A \leq U$ and $B \leq V$, thus $U^c \leq A^c$ and $V^c \leq B^c$

Since $A \leq V^c \Rightarrow \tau_i - \text{cl}^*(A) \leq \tau_i - \text{cl}^*(V^c) = V^c$

Because $V^c$ is $* - \tau_i - \text{IFCS}$

$\Rightarrow \tau_i - \text{cl}^*(A) \leq V^c \leq B^c$, since $B \leq U^c$

$\Rightarrow \tau_i - \text{cl}(B) \leq \tau_i - \text{cl}(U^c) = U^c$, because $U^c$ is $\tau_i - \text{IFCS}$, $i \in \{1,2\}$

Thus $\tau_i - \text{cl}(B) \leq U^c \leq A^c$

By proposition (2.9) $\tau_i - \text{cl}^*(A) \leq B^c$,

Then $\tau_i - \text{cl}^*(A) \tilde{q} B \ldots (1)$

$\tau_i - \text{cl}(B) \leq A^c \Rightarrow \tau_i - \text{cl}(B) \tilde{q} A$, then $A \tilde{q} \tau_i - \text{cl}(B) \ldots (2)$

Hence $A$, $B$ are $\tau_i - \text{IF} * - \text{separated sets}$

**Definition 3.1.10:**

An $\tau_i - \text{intuitionistic fuzzy set}$ $E$ in intuitionistic fuzzy ideal bitopological space $(X, \tau_1, \tau_2, L)$ is said to be intuitionistic fuzzy $* - \text{connected}$ if it
can not be expressed as the Union of two intuitionistic fuzzy \( * \) - separated sets. Otherwise, \( E \) is said to be intuitionistic fuzzy \( * \) - disconnected. If \( E = X \), then \( X \) is said to be intuitionistic fuzzy \( * \) - connected space. And we shall denoted it by \( (\tau_1, \tau_2)_IF \), for short \( i \in \{1,2\} \).

**Theorem 3.1.11 :-**

Let \( A \) and \( B \) be \( \tau_i - \)intuitionistic fuzzy \( * \) - separated sets in an intuitionistic fuzzy ideal bitopological pace \((X, \tau_1, \tau_2, L)\) and \( E \) be a non empty \( \tau_i - \)IF \( * \) - connected set in \( X \) such that \( E \subseteq A \cup B \). Then exactly one of the following conditions holds:

a) \( E \leq A \) and \( E \cap B = 0 \_

b) \( E \leq B \) and \( \cap A = 0 \_

**Proof :-**

Let \( E \cap B = 0 \_

Since \( E \leq A \cup B \) then \( E \leq A \)

Similarly, if \( E \cup A = 0 \_

we have \( E \leq B \)

Since \( E \leq A \cup B \) then \( E \cap A = 0 \_

and \( E \cap B = 0 \_

can not hold simultaneously (because \( E \neq 0 \_

Suppose that \( E \cap B \neq 0 \_

and \( \cap A \neq 0 \_

Then \( E \cap A \) and \( E \cap B \) are \( \tau_i - \)IF \( * \) - separated set in \( X \) such that

\( E = (E \cap A) \cup (E \cap B) \) therefore \( E \) is an \( \tau_i - \)intuitionistic fuzzy \( * \) - disconnectedness of \( E \).

This is contradiction

Hence exactly one of the conditions (a) and (b) must hold.
Theorem 3.1.12 :-

Let $E,F$ be two $\tau_i$–intuitionistic fuzzy sets of IFLBTS $(X,\tau_1,\tau_2,L)$ if $E$ is an $\tau_i$–IF *–connected and $E \leq F \leq \tau_i – \text{cl}^*(E)$, $i \in \{1,2\}$. Then $F$ is an $\tau_i$–IF *–connected set.

Proof :-

If $= 0_\sim$, then the result is true.

Let $F \neq 0_\sim$ and $F$ is a IF *–disconnected. There exist two $\tau_i$–IF *–separated sets $A$ and $B$ in $X$ such that $F = \lor B$. Since $E$ is an $\tau_i$–IF *–connected and

$E \leq F = E \lor F, E \leq F = A \lor B, E \leq A \lor B$

So by theorem ( 3.1.11 ), we get

$E \leq A$ and $E \land B = 0_\sim$ or $E \leq B$ and $E \land A = 0_\sim$

Let $E \leq A$ and $E \land B = 0_\sim$

$B = B \land F \leq B \land \tau_i – \text{cl}^*(E) \leq B \land \tau_i – \text{cl}^*(A) \leq B \land B^c \leq B$, $i \in \{1,2\}$

It follows that $B = B \land B^c$ when $B = 0_\sim$ or $\mu_B(x) = \nu_B(x), \forall x \in X$.

Since $\neq 0_\sim \implies \mu_B(x) = \nu_B(x), \forall x \in X$.

Thus, $B_r = X$ where $B_0$ denotes the support of $B$.

Now $E \land B = 0_\sim$ implies $E_r \land B_r = \emptyset \implies E_r = \emptyset \implies E = \emptyset$

Which is a contradiction

Similarly, if $E \leq B$ and $E \land A = 0_\sim$, then we get $E = 0_\sim$ a contradiction

Hence $F$ is an $\tau_i$–intuitionistic fuzzy *–connected.

Theorem 3.1.13 :-

Let $A$ and $B$ be two $\tau_i$–intuitionistic fuzzy *–connected sets which are not $\tau_i$–intuitionistic fuzzy *–separated. Then $A \lor B$ is $\tau_i$–intuitionistic fuzzy*–connected set.
**Proof :**

Suppose that $A \lor B$ is an $\tau_i$—intuitionistic fuzzy *—disconnected set $\implies A \lor B = G \lor H$ where $G$ and $H$ are $\tau_i$—intuitionistic fuzzy *—separated sets in $X$.

Since $A \leq A \lor B$ and $B \leq A \lor B$

Then $A \leq G \lor H$ and $B \leq G \lor H$

By theorem (3.1.11), we get

$A \leq G$ with $A \land H = 0_\sim$ or $A \leq H$ with $A \land G = 0_\sim$.  

And $B \leq G$ with $B \land H = 0_\sim$ or $B \leq H$ with $\land G = 0_\sim$.  

If $A \leq G$ and $B \leq H$ or $A \leq H$ and $B \leq G$

We get that $A$ and $B$ are $\tau_i$—intuitionistic fuzzy *—separated and this contradiction

If $A \leq G$ with $B \land H = 0_\sim$ and $B \leq G$ with $\land H = 0_\sim$.

If $A \leq H$ with $A \land G = 0_\sim$ and $B \leq H$ with $B \land G = 0_\sim$

We get that

$A \lor B \leq G$ and $H = 0_\sim$ or $A \lor B \leq H$ and $G = 0_\sim$ which contradiction, therefore, $A \lor B$ is $\tau_i$—intuitionistic fuzzy *—connected set.

**Theorem 3.1.14 :**

Let $f: (X, \tau_1, \tau_2, L) \rightarrow (Y, \tau_1, \tau_2)$ is intuitionistic fuzzy continuous on to mapping, if $(X, \tau_1, \tau_2, L)$ is an $\tau_i$—intuitionistic fuzzy *—connected ideal bitopological space. Then $(Y, \tau_1, \tau_2)$ is also $\tau_i$—intuitionistic fuzzy *—connected bitopological space.

**Proof :**

It is known that connectedness is preserved by intuitionistic fuzzy continuous surjections.
The proof is clear.

**Corollary 3.1.15** :-

If IFS $A$ is an $\tau _1$—intuitionistic fuzzy $*-$connected set in an intuitionistic fuzzy ideal bitopological space $(X, \tau _1, \tau _2, L)$ . Then $\tau _i - cl^*(A)$ , $i \in \{1,2\}$ is $\tau _i$—intuitionistic fuzzy $*-$connected set .

**Proof :-**

Since $\tau _i - cl^*(A) = A \cup A^*(L, \tau _i), i \in \{1,2\}$ ,

Then $\subseteq \tau _i - cl^*(A)$ .

Since $A$ is $\tau _i$—IF $*-$connected set and $A \subseteq \tau _i - cl^*(A)$ .

By theorem ( 3.1.12 )

$\tau _i - cl^*(A)$ is an $\tau _i$—IF $*-$connected set .

**Theorem 3.1.16** :-

If $\{\mu _i; i \in N\}$ is a non empty family of $\tau _i$—intuitionistic fuzzy $*-$connected sets of an IFLBTS $(X, \tau _1, \tau _2, L)$ with $\bigcap _{i \in I} \mu _i \neq \emptyset$ . Then $\bigcup _{i \in I} \mu _i$ is an $\tau _i$—intuitionistic fuzzy $*-$connected set .

**Proof :-**

Suppose that $\bigcup _{i \in I} \mu _i$ is not $\tau _i$—IF $*-$connected set .

Then by definition ( 3.1.10 ) , there exist two $\tau _i$—IF $*-$separated sets $H$ and $G$ , such that

$\bigcup _{i \in I} \mu _i = H \cup G$ , since $\bigcap _{i \in I} \mu _i \neq \emptyset$ . We have a point $x$ in $\bigcap _{i \in I} \mu _i$ .

Since $\epsilon \bigcup _{i \in I} \mu _i$ , either $x \in H$ or $x \in G$ .

Suppose that $\epsilon X$ . Since $x \in \mu _i$ for each $\epsilon N$ , then $\mu _i$ and $H$ intersect for each $i \epsilon N$ .

By theorem ( 3.1.11 ) $\mu _i \subseteq H$ and $\mu _i \wedge G = 0_-$ or $\mu _i \subseteq G$ and $\mu _i \cap H = 0_-$ .

Suppose that $\mu _i \subseteq H \Rightarrow \mu _i \subseteq H$ for all $i \epsilon N$ and hence $\bigcup _{i \epsilon I} \mu _i \subseteq H$ .

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This implies that $\tau_i -\text{IF} \ast -\text{separated set} \ G$ is empty.
This is a contradiction.
Suppose that $\mu_i \subset G$. By similar way, we get $H = \emptyset$.
And this is a contradiction.
Thus, $\cup_{i \in I} \mu_i$ is an $\tau_i -\text{intuitionistic fuzzy} \ast -\text{connected set}$.

**Theorem 3.1.17 :-**

Suppose that $\{\mu_n; n \in \mathbb{N}\}$ is an sequence of $\tau_i -\text{intuitionistic fuzzy} \ast -\text{connected open sets}$ of an intuitionistic fuzzy ideal bitopological space $(X, \tau_1, \tau_2, L)$ and $\mu_n \cap \mu_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$. Then $\cup_{i \in I} \mu_i$ is $\tau_i -\text{IF} \ast -\text{connected set}$.

**Proof :-**
By induction and theorem (3.1.16)
The $N_n = \cup_{k \leq n} \mu_k$ is $\tau_i -\text{IF} \ast -\text{connected open set}$ for each $n \in \mathbb{N}$
Also, $N_n$ is $\tau_i -\text{IF} \ast -\text{connected open set}$.
Thus, $\cup_{n \in \mathbb{N}} \mu_n$ is $\tau_i -\text{IF} \ast -\text{connected set}$.

**References**


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